

ALGEBRAIC ANABELIAN FUNCTORS

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ABSTRACT. In this paper we will prove that there exists a covariant functor, called algebraic anabelian functor, from the category of algebraic schemes over a given field to the category of outer homomorphism sets of groups. The algebraic anabelian functor, given in a canonical manner, is full and faithful. It reformulates the anabelian geometry over a field. As an application of the anabelian functor, we will also give a proof of the section conjecture of Grothendieck for the case of algebraic schemes.

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INTRODUCTION

In this paper we will prove that there exists an *algebraic anabelian functor*, a covariant functor, given in a canonical manner, from the category of algebraic schemes over a given field to the category of outer homomorphism sets of groups.

Fortunately, the algebraic anabelian functor is full and faithful. In deed, it reformulates the anabelian geometry over a field in the sense

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of Grothendieck. For detail, see *Theorem 1.3*, the main theorem of the paper.

As an application of the algebraic anabelian functor, we will give a proof of the section conjecture of Grothendieck for the case of algebraic schemes. See §2 for detail.

We will prove the Main Theorem of the paper in §12 after we make several preparations in §§3-11.

In particular, in §9 we will give the proofs that the arithmetic unramified extension in [5] and the formally unramified extension in [7] are both well-defined. These unramified extensions are used to give the computations of étale fundamental groups for arithmetic schemes and algebraic schemes, respectively.

Note that there exists another anabelian functor, the **arithmetic anabelian functor**, which is a covariant functor defined canonically on the category of arithmetic schemes surjectively over the ring \mathcal{O}_K of algebraic integers of a number field K . Such a functor is also full and faithful.

However, the arithmetic anabelian functors, which are related to class field theory, are very different from the algebraic ones. This is due to the fact that their étale fundamental groups are very different. For example, it is clear that

$$\pi_1^{et}(Spec(\mathbb{Q})) \cong Gal(\overline{\mathbb{Q}}/\mathbb{Q})$$

holds (or see *Theorem 10.2* for a generalized result). On the other hand, by [5, 9] we have

$$\pi_1^{et}(Spec(\mathbb{Z})) \cong Gal(\mathbb{Q}^{un}/\mathbb{Q}) = \{0\}.$$

Here, \mathbb{Q}^{un} ($= \mathbb{Q}$) denotes the (nonabelian) maximal unramified extension of the rational field \mathbb{Q} .

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1. STATEMENT OF THE MAIN THEOREM

1.1. Notation. Let K be a field. By an **algebraic K -variety** we will understand an integral scheme X over K of finite type.

For an integral scheme Z , put

- $k(Z) \triangleq \mathcal{O}_{X,\xi}$, the function field of an integral scheme Z with generic point ξ ;
- $\pi_1^{et}(Z) \triangleq$ the étale fundamental group of Z for a geometric point of Z over a separable closure of the function field $k(Z)$.

In particular, for a field L , set

$$\pi_1^{et}(L) \triangleq \pi_1^{et}(\text{Spec}(L)).$$

1.2. Outer homomorphism set. Let G, H, π_1, π_2 be four groups with homomorphisms $p : G \rightarrow \pi_1$ and $q : H \rightarrow \pi_2$, respectively.

Definition 1.1. The **outer homomorphism set** from G into H over π_1 and π_2 respectively, denoted by $\text{Hom}_{\pi_1, \pi_2}^{out}(G, H)$, is the set of the maps σ from the quotient $\frac{\pi_1}{p(G)}$ into the quotient $\frac{\pi_2}{q(H)}$, given by a group homomorphism $f : G \rightarrow H$ in such a manner:

$$\sigma : \frac{\pi_1}{p(G)} \rightarrow \frac{\pi_2}{q(H)}, x \cdot p(G) \mapsto f(x) \cdot q(H)$$

for any $x \in \pi_1$.

In particular, such a σ is said to be **bijective** if the f above is an isomorphism such that $q(H) = f \circ p(G)$.

Remark 1.2. Suppose that G and H are normal subgroups of π_1 and π_2 , respectively. Then $\text{Hom}_{\pi_1, \pi_2}^{out}(G, H)$ can be regarded as a subset of $\text{Hom}(\text{Out}(G), \text{Out}(H))$.

Let $\mathbb{P}\mathbb{G}$ be the category of group pairs (G, π) as objects, with outer homomorphisms between group pairs as morphisms. Here, By a **group pair** (G, π) we understand two groups G and π together with a given homomorphism $p : G \rightarrow \pi$ of groups. By an **outer homomorphism** from (G, π_1) into (H, π_2) we understand a map

$$\sigma : \frac{\pi_1}{p(G)} \rightarrow \frac{\pi_2}{q(H)}$$

given in *Definition 1.1*.

$\mathbb{P}\mathbb{G}$ will be called the **category of outer homomorphism sets of groups** in the paper.

1.3. Statement of the main theorem. Fixed a field K . Here K is not necessarily of characteristic zero. For any algebraic K -variety X , there canonically exists a group pair $(\pi_1^{et}(X), \pi_1^{et}(k(X)))$.

Let $\text{Sch}(K)$ denote the category of algebraic K -varieties as objects, with scheme morphisms as morphisms satisfying the condition:

For any $X, Y \in \text{Sch}(K)$, a scheme morphism $f : X \rightarrow Y$ is said to be contained in the category $\text{Sch}(K)$ if $k(X)$ is separably generated over $k(Y)$ canonically.

Here is the main theorem of the present paper.

Theorem 1.3. (Main Theorem) *For any field K , there is a covariant functor τ from category $\mathcal{Sch}(K)$ to category \mathbb{PG} given in a canonical manner:*

- *An algebraic K -variety $X \in \mathcal{Sch}(K)$ is mapped into a group pair $(\pi_1^{et}(X), \pi_1^{et}(k(X))) \in \mathbb{PG}$;*
- *A scheme morphism $f \in \text{Hom}(X, Y)$ is mapped into an outer homomorphism $\tau(f) \in \text{Hom}_{\pi_1^{et}(k(X)), \pi_1^{et}(k(Y))}^{out}(\pi_1^{et}(X), \pi_1^{et}(Y))$ given by f .*

Furthermore, τ is full and faithful.

In particular, an $f \in \text{Hom}(X, Y)$ is an isomorphism if and only if X and Y have a common sp -completion and $\tau(f)$ is a bijective outer homomorphism.

Here, for sp -completion, see [9] or see below §8.2 in the present paper. Roughly speaking, an sp -completion of an integral scheme is such a one that contains all the separably closed points.

We will prove *Theorem 1.3* in §12 after we make preparations in §§3-11.

Remark 1.4. The functor $\tau(K)$ is said to be the **anabelian functor** over a field K , or **algebraic anabelian functor**. The Main Theorem above says that the anabelian functor over a field reformulates the anabelian geometry in the sense of Grothendieck. In deed, it will also give an answer to the section conjecture of Grothendieck for the case of algebraic schemes (see §2).

Remark 1.5. There exists an **arithmetic anabelian functor**, which is the anabelian functor over the ring \mathcal{O}_K of algebraic integers of a number field K . Such a functor is also full and faithful. However, the arithmetic anabelian functors, which are related to class field theory, are very different from the algebraic ones. This is due to the fact that their étale fundamental groups are very different.

Remark 1.6. Let X and Y be two integral K -varieties. It is seen that the group pairs $(\pi_1^{et}(X), \pi_1^{et}(k(X)))$ and $(\pi_1^{et}(Y), \pi_1^{et}(k(Y)))$ are indeed the ramified groups $\pi_1^{br}(X)$ and $\pi_1^{br}(Y)$, respectively. Hence, the outer homomorphism set

$$\text{Hom}_{\pi_1^{et}(k(X)), \pi_1^{et}(k(Y))}^{out}(\pi_1^{et}(X), \pi_1^{et}(Y))$$

is exactly equal to the set

$$\text{Hom}(\pi_1^{br}(X), \pi_1^{br}(Y))$$

of homomorphisms between the ramified groups. For ramified groups, see §10-11 below in the paper.

2. APPLICATION TO THE SECTION CONJECTURE

2.1. Algebraic anabelian functor: Special case. Let $\mathcal{S}\mathcal{h}(K)_0$ be the category of algebraic K -varieties as objects, together with scheme morphisms as morphisms satisfying the condition:

For any $X, Y \in \mathcal{S}\mathcal{h}(K)_0$, a morphism $f : X \rightarrow Y$ of schemes is said to be contained in the category $\mathcal{S}\mathcal{h}(K)_0$ if X and Y have a common sp-completion and $k(X)$ is separable over $k(Y)$ canonically.

Here is a result on algebraic anabelian functor on the category $\mathcal{S}\mathcal{h}(K)_0$.

Theorem 2.1. *For any field K , there exists a covariant functor τ_0 from category $\mathcal{S}\mathcal{h}(K)_0$ to category $\mathbb{P}\mathbb{G}$ given in a canonical manner:*

- *An algebraic K -variety $X \in \mathcal{S}\mathcal{h}(K)_0$ is mapped into a group pair $(\pi_1^{et}(X), \pi_1^{et}(k(X))) \in \mathbb{P}\mathbb{G}$;*
- *A scheme morphism $f \in \text{Hom}(X, Y)$ is mapped into an outer homomorphism $\tau_0(f) \in \text{Hom}_{\pi_1^{et}(k(X)), \pi_1^{et}(k(Y))}^{out}(\pi_1^{et}(X), \pi_1^{et}(Y))$ given by f .*

Furthermore, τ_0 is full and faithful.

In particular, an $f \in \text{Hom}(X, Y)$ is an isomorphism if and only if the outer homomorphism $\tau_0(f)$ is bijective.

Proof. It is immediate from *Theorem 1.3* and *Lemma 12.2*. □

2.2. Section conjecture for algebraic schemes. Fixed a field K . There are the following results on anabelian geometry of algebraic schemes over K .

Theorem 2.2. *Let X and Y be two algebraic K -varieties. Suppose that $k(X)$ is canonically separable over $k(Y)$ and X, Y have a common sp-completion. Then there is a bijection*

$$\text{Hom}(X, Y) \cong \text{Hom}_{\pi_1^{et}(k(X)), \pi_1^{et}(k(Y))}^{out}(\pi_1^{et}(X), \pi_1^{et}(Y))$$

between sets.

Proof. It is immediate from *Theorem 2.1*. □

Theorem 2.3. *Let X be an algebraic K -variety. Suppose that $k(X)$ is separably generated over K . Then there is a bijection*

$$\Gamma(X/K) \cong \text{Hom}_{\pi_1^{et}(K), \pi_1^{et}(k(X))}^{out}(\pi_1^{et}(K), \pi_1^{et}(X))$$

between sets.

Proof. It is immediate from *Theorem 1.3* and *Lemma 12.4*. □

2.3. An interpretation given by Galois groups. For a field L , set the following symbols

- $G(L) \triangleq$ the absolute Galois group $Gal(L^{sep}/L)$;
- $G(L)^{au} \triangleq$ the Galois group $Gal(L^{au}/L)$ of the maximal formally unramified extension L^{au} of L (see *Definition 9.7* below).

For *Theorems 2.2-3*, there is the following version of Galois groups of fields.

Theorem 2.4. *Let X and Y be two algebraic K -varieties. Suppose that $k(X)$ is canonically separable over $k(Y)$ and X, Y have a common sp -completion. Then there is a bijection*

$$Hom(X, Y) \cong Hom_{G(k(X)), G(k(Y))}^{out} (G(k(X))^{au}, G(k(Y))^{au})$$

between sets.

Theorem 2.5. *Let X be an algebraic K -variety. Suppose that $k(X)$ is separably generated over K . Then there is a bijection*

$$\Gamma(X/K) \cong Hom_{G(K), G(k(X))}^{out} (G(K)^{au}, G(k(X))^{au})$$

between sets.

It is seen that *Theorems 2.4-5* hold from *Theorems 2.2-3* above.

3. BASIC DEFINITIONS

Let's fix notation and terminology in the present paper. They will be used in the following sections.

3.1. Convention. For an integral domain D , let $Fr(D)$ denote the field of fractions of D .

In particular, let D be contained in a field Ω . In the paper $Fr(D)$ will always be assumed to be contained in Ω .

For a field L , set

- $L^{sep} \triangleq$ the separable closure of L ;
- L^{al} (or \bar{L}) \triangleq an algebraic closure of L .

By an **integral K -variety** X in the paper we will understand an integral scheme over a field K (not necessarily of finite type).

3.2. Quasi-galois extension of a function field. Assume that L is an extension over a field K . Let $Gal(L/K)$ be the Galois group of L over K . Note that here L is not necessarily algebraic over K .

Definition 3.1. The field L is said to be **Galois** over K if K is the invariant subfield of $Gal(L/K)$.

For example, $\mathbb{Q}(t)$ is Galois over \mathbb{Q} . Here, t is a variable over \mathbb{Q} .

Now we extend the notion of quasi-galois from algebraic extensions to function fields.

Definition 3.2. The field L is said to be **quasi-galois** over K if each irreducible polynomial $f(X) \in F[X]$ that has a root in L factors completely in $L[X]$ into linear factors for any subfield F with $K \subseteq F \subseteq L$.

Let $D \subseteq D_1 \cap D_2$ be three integral domains. Then D_1 is said to be **quasi-galois** over D if the fraction field $Fr(D_1)$ is quasi-galois over $Fr(D)$.

Definition 3.3. The ring D_1 is said to be a **conjugation** of D_2 over D if there is an F -isomorphism $\tau : Fr(D_1) \rightarrow Fr(D_2)$ such that $\tau(D_1) = D_2$, where $F \triangleq k(\Delta)$, $k \triangleq Fr(D)$, Δ is a transcendental basis of the field $Fr(D_1)$ over k , and F is contained in $Fr(D_1) \cap Fr(D_2)$.

In such a case, D_1 is also said to be a D -**conjugation** of D_2 .

Replacing rings by fields, we have a definition that a field L_1 is said to be a **conjugation** of a field L_2 over a field K , where K is assumed to be contained in the intersection $L_1 \cap L_2$. Note that in such a case, we must have $\overline{L_1} = \overline{L_2}$.

3.3. Essentially affine scheme. Let X be a scheme. As usual, an **affine covering** of X is a family $\mathcal{C}_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ such that for each $\alpha \in \Delta$, ϕ_α is an isomorphism from scheme $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ onto scheme $(Spec(A_\alpha), \mathcal{O}_{Spec(A_\alpha)})$, where A_α is a commutative ring with identity.

Each element $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ is called a **local chart**. For the sake of brevity, a local chart $(U_\alpha, \phi_\alpha; A_\alpha)$ will be denoted by U_α or (U_α, ϕ_α) .

An affine covering \mathcal{C}_X of (X, \mathcal{O}_X) is said to be **reduced** if $U_\alpha \neq U_\beta$ holds for any $\alpha \neq \beta$ in Δ .

Definition 3.4. An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of X is said to be an **affine patching** of X if $U_\alpha = Spec A_\alpha$ and the map ϕ_α is the identity map on the underlying space U_α for each $\alpha \in \Delta$.

Let \mathfrak{Comm} be the category of commutative rings with identity. For a given field Ω , let $\mathfrak{Comm}(\Omega)$ be the category consisting of the subrings of Ω and their isomorphisms.

Definition 3.5. Let \mathfrak{Comm}_0 be a subcategory of \mathfrak{Comm} . An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of X is said to be **with values** in \mathfrak{Comm}_0 if for each $\alpha \in \Delta$ there are $\mathcal{O}_X(U_\alpha) = A_\alpha$ and $U_\alpha = Spec(A_\alpha)$, where A_α is a ring contained in \mathfrak{Comm}_0 .

In particular, an affine covering \mathcal{C}_X of X with values in $\mathfrak{Comm}(\Omega)$ is said to be **with values** in the field Ω .

Let $\mathcal{C}_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ be a reduced affine covering of X with values in a field Ω . For any $(U_\alpha, \phi_\alpha; A_\alpha), (U_\beta, \phi_\beta; A_\beta) \in \mathcal{C}_X$, we say

$$(U_\alpha, \phi_\alpha; A_\alpha) = (U_\beta, \phi_\beta; A_\beta)$$

if and only if

$$U_\alpha = U_\beta, \phi_\alpha = \phi_\beta.$$

That is, we will always neglect the map ϕ_α for a local chart $(U_\alpha, \phi_\alpha; A_\alpha)$ in such a \mathcal{C}_X .

For brevity, a scheme is said to be **essentially affine** in Ω if it has a reduced affine covering with values in Ω .

It will be seen that essentially affine schemes have many properties like affine schemes.

3.4. Essentially equal scheme. By affine covering with values in a field, it is seen that affine open sets in a scheme is measurable and the non-affine open sets are unmeasurable. So we can neglect the non-affine open sets in an evident manner, where almost every property of the scheme is preserved.

Now suppose that there are two structure sheaves \mathcal{O}_X and \mathcal{O}'_X on the underlying space of an integral scheme X .

Definition 3.6. The two integral schemes (X, \mathcal{O}_X) and (X, \mathcal{O}'_X) are said to be **essentially equal** provided that for any open set U in X , there is an equivalence relation

$$U \text{ is affine open in } (X, \mathcal{O}_X) \iff \text{so is } U \text{ in } (X, \mathcal{O}'_X)$$

and in such a case, either $D_1 = D_2$ holds or the two conditions below are both satisfied¹:

- $Fr(D_1) = Fr(D_2)$.
- For any nonzero $x \in Fr(D_1)$, there is a relation

$$x \in D_1 \bigcap D_2$$

or there is an equivalence relation

$$x \in D_1 \setminus D_2 \iff x^{-1} \in D_2 \setminus D_1.$$

Here, $D_1 = \mathcal{O}_X(U)$ and $D_2 = \mathcal{O}'_X(U)$.

For example, consider the discrete valuation ring

$$\mathbb{Z}_{(p)} = \left\{ \frac{r}{s} : r, s \in \mathbb{Z}, s \neq 0, (p, s) = 1 \right\}$$

¹By such additional conditions, we have a sufficiently large number of integral schemes so that we can give a computation of étale fundamental groups.

of $\text{Spec}(\mathbb{Q})$ for a prime p . It is clear that $\text{Spec}(\mathbb{Z}_{(3)})$ and $\text{Spec}(\mathbb{Z}_{(5)})$ are not essentially equal.

In deed, let $\dim X = 1$. Suppose that the integral schemes (X, \mathcal{O}_X) and (X, \mathcal{O}'_X) are essentially equal. Then $\mathcal{O}_X(U)$ and $\mathcal{O}'_X(U)$ have the same discrete valuation for an affine open set U in X .

Definition 3.7. Any two schemes (X, \mathcal{O}_X) and (Z, \mathcal{O}_Z) are said to be **essentially equal** if the underlying spaces of X and Z coincide with each other and the schemes (X, \mathcal{O}_X) and (X, \mathcal{O}_Z) are essentially equal.

It is seen that scheme that are essentially equal must be isomorphic.

3.5. Quasi-galois closed affine covering. Assume that $f : X \rightarrow Y$ is a surjective morphism between integral schemes. Fixed an algebraic closure Ω of the function field $k(X)$.

Definition 3.8. A reduced affine covering \mathcal{C}_X of X with values in Ω is said to be **quasi-galois closed** over Y by f if there exists a local chart $(U'_\alpha, \phi'_\alpha; A'_\alpha) \in \mathcal{C}_X$ such that $U'_\alpha \subseteq \varphi^{-1}(V_\alpha)$ holds

- for any affine open set V_α in Y ;
- for any $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ with $U_\alpha \subseteq f^{-1}(V_\alpha)$;
- for any conjugate A'_α of A_α over B_α ,

where B_α is the canonical image of $\mathcal{O}_Y(V_\alpha)$ in $k(X)$ via f .

3.6. Quasi-galois closed scheme. Let X and Y be integral schemes. Suppose that $f : X \rightarrow Y$ is a surjective morphism. Denote by $\text{Aut}(X/Y)$ the group of automorphisms of X over Y .

An integral scheme Z is said to be a **conjugate** of X over Y if there is an isomorphism $\sigma : X \rightarrow Z$ over Y .

Definition 3.9. The scheme X is said to be **quasi-galois closed** (or **qc** for short) over Y by f if there is an algebraically closed field Ω and a reduced affine covering \mathcal{C}_X of X with values in Ω such that for any conjugate Z of X over Y the two conditions are both satisfied:

- (X, \mathcal{O}_X) and (Z, \mathcal{O}_Z) are essentially equal if Z is essentially affine in Ω .
- $\mathcal{C}_Z \subseteq \mathcal{C}_X$ holds if \mathcal{C}_Z is a reduced affine covering of Z with values in Ω .

Remark 3.10. In the above definition, the field Ω enables the affine open subschemes of the integral scheme X to be *measurable* while the other open subschemes of X are still *unmeasurable*. In particular, each scheme X that is *qc* over Y must be essentially affine.

Remark 3.11. Let Ω be the algebraically closed field in *Definition 3.9*.

(i) By Ω , all the rings of affine open sets in X are taken to be as subrings of the same ring Ω so that they can be compared with each other.

(ii) By Ω , we can restrict ourselves only to consider the function fields which have the same variables over a given field.

Remark 3.12. It is seen that in *Definition 3.9*, the affine covering \mathcal{C}_X of X is maximal by set inclusion. In fact, \mathcal{C}_X is the *natural affine structure* of X with values in Ω (see [2] for definition). Conversely, it can be proved that a quasi-galois closed scheme has a unique natural affine structure with values in Ω (see [2, 8]).

In other words, Ω can be chosen to be an algebraic closure of the function field $k(X)$; \mathcal{C}_X is the unique maximal affine covering of X with values in Ω (see *Remark 5.2* in §5 below).

4. GALOIS EXTENSIONS OF FUNCTION FIELDS

4.1. Galois extension of a function field. In this section we will prove the following result about a Galois extension of a function field.

Theorem 4.1. *Let L be a finitely generated extension of a field K . Then L is Galois over K if and only if L is quasi-galois and separably generated over K .*

Note that here L is not necessarily algebraic over K . After several lemmas, we will prove *Theorem 4.1* at the end of the present section.

4.2. Recalling preliminary facts on quasi-galois extensions. Let L be a finitely generated extension of a field K .

The elements $w_1, w_2, \dots, w_n \in L$ are said to be a (r, n) -**nice basis** of L over K if the conditions below are satisfied:

- $L = K(w_1, w_2, \dots, w_n)$;
- w_1, w_2, \dots, w_r are a transcendental basis of L over K ;
- $w_{r+1}, w_{r+2}, \dots, w_n$ are a linear basis of L over $K(w_1, w_2, \dots, w_r)$.

Here $0 \leq r \leq n$.

Let's recall preliminary facts on quasi-galois extensions of function fields.

Lemma 4.2. ([2, 3]) *Fixed an intermediate field $K \subseteq F \subsetneq L$ and an $x \in L$ that is algebraic over F . Let z be a conjugate of x over F . Then there exists a (s, m) -nice basis v_1, v_2, \dots, v_m of L over $F(x)$ and an F -isomorphism τ from the field*

$$L = F(x, v_1, v_2, \dots, v_s, v_{s+1}, \dots, v_m)$$

onto a field of the form

$$F(z, v_1, v_2, \dots, v_s, w_{s+1}, \dots, w_m)$$

such that

$$\tau(x) = z, \tau(v_1) = v_1, \dots, \tau(v_s) = v_s.$$

Here, $w_{s+1}, w_{s+2}, \dots, w_m$ are elements contained in an extension of F .

In particular, we have

$$w_{s+1} = v_{s+1}, w_{s+2} = v_{s+2}, \dots, w_m = v_m$$

if z is not contained in $F(v_1, v_2, \dots, v_m)$.

Proof. It is immediate from preliminary facts on fields. \square

Lemma 4.3. ([2, 3]) *The following statements are equivalent.*

- (i) L is quasi-galois over K .
- (ii) Any conjugation of L over K is contained in L .
- (iii) There exists one and only one conjugation of L over K .
- (iv) Take any $x \in L$ and any subfield $K \subseteq F \subseteq L$. Then L contains all conjugations of $F(x)$ over F .

Proof. Prove (i) \implies (iv). Fixed an $x \in L$ and a subfield $K \subseteq F \subseteq L$. If x is a variable over F , the field $F(x)$ must be contained in L since $F(x)$ is the unique conjugation of $F(x)$ over F by (i).

Let x be algebraic over F . Then an F -conjugation of $F(x)$, which is exactly an F -conjugate of $F(x)$, must be contained in L by (i).

Prove (iv) \implies (i). Fixed any subfield $K \subseteq F \subseteq L$. Let $f(X)$ be an irreducible polynomial over F . Take any $x \in L$ such that $f(x) = 0$. It is seen that an F -conjugation of $F(x)$ is nothing other than an F -conjugate. Then every F -conjugate of $F(x)$ is contained in L by (iv). Hence, L is quasi-galois over K .

Prove (iv) \implies (ii). Let H be a conjugation of L over K . Fixed any $x_0 \in H$. Take a (r, n) -nice basis w_1, w_2, \dots, w_n of L over K and an isomorphism $\sigma : H \rightarrow L$ over

$$K_0 \triangleq K(w_1, w_2, \dots, w_r)$$

such that H is a K -conjugation of L by σ .

It is clear that w_1, w_2, \dots, w_r are all contained in the intersection of H and L . It follows that x_0 must be algebraic over K_0 .

Evidently, the field $K_0[x_0]$ is a conjugate of the field $K_0[\sigma(x_0)]$ over K_0 and then is a conjugation of $K_0[\sigma(x_0)]$ over K . From (iv) we have $x_0 \in K_0[x_0] \subseteq L$. Hence, $H \subseteq L$.

Prove (ii) \implies (iv). Take any $x \in L$ and any subfield $K \subseteq F \subseteq L$.

If x is a variable over F , the field $F(x)$ that is the unique conjugation of $F(x)$ itself over F must be contained in L by (ii).

Suppose that x is algebraic over F . Let z be an F -conjugate of x . If $F = L$, we have $z = x \in L$ by (ii).

Now let $F \neq L$. From *Lemma 4.2* we have a field of the form

$$F(z, v_1, v_2, \dots, v_s, w_{s+1}, \dots, w_m),$$

which is an F -conjugation of L . As $K \subseteq F$, we must have

$$z \in F(z, v_1, v_2, \dots, v_s, w_{s+1}, \dots, w_m) \subseteq L.$$

by (ii) again. Hence, $z \in L$.

Prove (iii) \implies (ii). Trivial.

Prove (i) \implies (iii). Let L be quasi-galois over K and let H be a conjugation of L over K . In the following we will prove $H = L$.

In fact, choose a (s, m) -nice basis v_1, v_2, \dots, v_m of L over K and an F -isomorphism τ of H onto L such that via τ the field H is a conjugate of L over F , where

$$F \triangleq k(v_1, v_2, \dots, v_s).$$

It is seen that $F \subseteq H \subseteq L$ hold by *Definition 3.2*.

Hypothesize $H \subsetneq L$. Take any $x_0 \in L \setminus H$. There are two cases.

Case (i). Let x_0 be a variable over H . We have

$$\dim_K H = \dim_K L = s < \infty$$

since H and L are conjugations over K . On the other hand, we have

$$1 + \dim_K H = \dim_K H(x_0) \leq \dim_K L$$

from $x_0 \in L \setminus H$, which will be in contradiction.

Case (ii). Let x_0 be algebraic over H . It is seen that x_0 is algebraic over F . We have

$$[H : F] = [L : F] < \infty$$

since H is a conjugate of L over F . On the other hand, we have

$$2 + [H : F] \leq [H[x_0] : F] \leq [L : F]$$

from $x_0 \in L \setminus H$, which will be in contradiction.

Then $L \setminus H$ must be empty. Hence, $L = H$.

This completes the proof. \square

4.3. Proof of Theorem 4.1. Now we give the proof of *Theorem 4.1*:

Proof. Prove \Leftarrow . Let L be quasi-galois and separably generated over K . If L is algebraic over K , it is clear that L is Galois over K .

Now suppose that L is a transcendental extension over K . It suffices to prove that there exists an automorphism $\sigma_0 \in \text{Gal}(L/K)$ such that K is the invariant subfield of σ_0 .

In fact, fixed any (r, n) -nice basis v_1, v_2, \dots, v_n of L over K . Put

$$F_0 \triangleq K(v_1, v_2, \dots, v_r).$$

Then L is algebraic over F_0 . Here, we have $r \geq 1$. By *Lemma 4.3* it is seen that every conjugation of L over K is exactly L itself. It follows that there is one and only conjugate of L over F_0 . Then L is a quasi-galois algebraic extension of F_0 .

Hence, L is Galois over F_0 since L is separable over F_0 from the assumption. Fixed any $\tau_0 \in \text{Gal}(L/F_0)$ with $\tau_0 \neq \text{id}_L$.

Let τ_1 be an automorphism of F_0 over K given by

$$v_1 \mapsto \frac{1}{v_1}, v_2 \mapsto \frac{1}{v_2}, \dots, v_r \mapsto \frac{1}{v_r}.$$

Then we have an automorphism $\sigma_0 \in \text{Gal}(L/K)$ defined by τ_0 and τ_1 in such a manner

$$\begin{aligned} & \frac{f(v_1, v_2, \dots, v_n)}{g(v_1, v_2, \dots, v_n)} \in L \\ \mapsto & \frac{f(\tau_1(v_1), \tau_1(v_2), \dots, \tau_1(v_r), \tau_0(v_{r+1}), \dots, \tau_0(v_n))}{g(\tau_1(v_1), \tau_1(v_2), \dots, \tau_1(v_r), \tau_0(v_{r+1}), \dots, \tau_0(v_n))} \in L \end{aligned}$$

for any polynomials $f(X_1, X_2, \dots, X_n)$ and $g(X_1, X_2, \dots, X_n) \neq 0$ over the field K with $g(v_1, v_2, \dots, v_n) \neq 0$.

It is easily seen that we have

$$g(v_1, v_2, \dots, v_n) = 0$$

if and only if

$$g(v_1, v_2, \dots, v_r, \tau_0(v_{r+1}), \dots, \tau_0(v_n)) = 0$$

if and only if

$$g(\tau_1(v_1), \tau_1(v_2), \dots, \tau_1(v_r), \tau_0(v_{r+1}), \dots, \tau_0(v_n)) = 0.$$

Hence, σ_0 is well-defined.

It is seen that K is the invariant subfield of the automorphism σ_0 of L over K . Hence, K is the invariant subfield of the Galois group $\text{Gal}(L/K)$. This proves that L is Galois over K .

Prove \implies . Let L is Galois over K . Fixed any (r, n) -nice basis of L over K , namely v_1, v_2, \dots, v_n . Set

$$F_0 \triangleq K(v_1, v_2, \dots, v_r).$$

By *Lemma 3.1* it is seen that the field F_0 must be invariant under the Galois group $\text{Gal}(L/F_0)$. It follows that L is a Galois algebraic extension over F_0 . Hence, L is separably generated over K .

Let H be a conjugate of L over F_0 . Then H is a conjugation of L over K . As L is Galois over F_0 , we have $H = L$; hence, there exists

one and only one conjugation of L over K under a (r, n) -nice basis of L over K .

Now let

$$v_1, v_2, \dots, v_n$$

run through all possible (r, n) -nice bases of L over K . It is seen that there exists one and only one conjugation of L over K . From *Lemma 4.3* it is immediate that L is quasi-galois over K . \square

5. KEY PROPERTY OF qc SCHEME

In this section we will prove a key property of a qc scheme and then give a criterion for such a scheme. By these results, we will obtain an essential and sufficient condition for a qc scheme.

5.1. Key property of a qc scheme. Let X and Y be integral schemes and let $f : X \rightarrow Y$ be a surjective morphism. We have the following key property for qc schemes.

Lemma 5.1. *Let Ω be an algebraic closure of the function field $k(X)$. Suppose that X is qc over Y by f . Then there is a unique maximal affine covering \mathcal{C}_X of X with values in Ω ; moreover, \mathcal{C}_X is quasi-galois closed over Y by f .*

Proof. From *Definition 3.9* we have an algebraically closed field Ω' and a reduced affine covering \mathcal{C}'_X of X with values in Ω' such that for any conjugate Z of X over Y the two conditions are satisfied:

- (X, \mathcal{O}_X) and (Z, \mathcal{O}_Z) are essentially equal if Z has a reduced affine covering with values in Ω' .
- $\mathcal{C}_Z \subseteq \mathcal{C}'_X$ holds if \mathcal{C}_Z is a reduced affine covering of Z with values in Ω' .

It is clear that Ω' contains the function field $k(X)$ and \mathcal{C}'_X is maximal by set inclusion.

Prove the uniqueness of \mathcal{C}'_X . In deed, let \mathcal{C}''_X be another reduced affine covering of X with values in Ω' satisfying the two conditions above. Then we must have $\mathcal{C}'_X = \mathcal{C}''_X$ according to the second condition.

Prove that \mathcal{C}'_X is quasi-galois closed over Y by f . In fact, take

- an affine open set V_α in Y ;
- a $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}'_X$ with $U_\alpha \subseteq f^{-1}(V_\alpha)$;
- a conjugate A'_α of A_α over B_α ,

where B_α is the canonical image of $\mathcal{O}_Y(V_\alpha)$ in $k(X)$ via f .

Then there must exist a local chart $(U'_\alpha, \phi'_\alpha; A'_\alpha) \in \mathcal{C}'_X$ such that $U'_\alpha \subseteq \varphi^{-1}(V_\alpha)$ holds; otherwise, if there is some $(U'_\alpha, \phi'_\alpha; A'_\alpha) \notin \mathcal{C}'_X$, we

will obtain a new reduced affine covering

$$\{(U'_\alpha, \phi'_\alpha; A'_\alpha)\} \bigcup \mathcal{C}'_X$$

of X , which is in contradiction to the uniqueness of \mathcal{C}'_X .

It follows that \mathcal{C}'_X is with values in an algebraic closure Ω of the function field $k(X)$ from *Definition 3.3*. \square

By *Lemma 5.1* we have the following remarks.

Remark 5.2. Let \mathcal{C}_X and Ω be assumed as in *Definition 3.9*. There are the following statements.

- The field Ω can be chosen to be an algebraic closure of the function field $k(X)$.
- The affine covering \mathcal{C}_X is quasi-galois closed over Y by f and is the unique maximal affine covering of X with values in Ω .

Remark 5.3. The affine covering \mathcal{C}_X above is the unique maximal affine structure of X with values in the algebraic closure Ω of $k(X)$. In [2], we use affine structures to get the key property as in *Lemma 5.1*.

5.2. Criterion for qc schemes. Fixed integral schemes X and Y . Let Ω be an algebraically closed closure of the function field $k(X)$.

Lemma 5.4. *Let $f : X \rightarrow Y$ be a surjective morphism of schemes. Then X is qc over Y by f if there is a unique maximal reduced affine covering \mathcal{C}_X of X with values in Ω such that \mathcal{C}_X is quasi-galois closed over Y by f .*

Proof. Assume that X has a unique maximal reduced affine covering \mathcal{C}_X with values in Ω that is quasi-galois closed over Y by f .

Fixed any conjugate Z of X over Y and any isomorphism $\sigma : Z \rightarrow X$ over Y . Suppose that Z has a reduced affine covering \mathcal{C}_Z with values in Ω .

Take a local chart $(W, \delta, C) \in \mathcal{C}_Z$. Put

$$U = \sigma(W); A = \mathcal{O}_X(U); C = \mathcal{O}_Z(W).$$

We have

$$U = \text{Spec}(A); W = \text{Spec}(C); A \subseteq \Omega; C \subseteq \Omega.$$

It is seen that there exists an affine open subset U' in X such that

$$C = \mathcal{O}_X(U')$$

by the assumption that \mathcal{C}_X is quasi-galois closed over Y . As

$$U' = \text{Spec}(C) = W,$$

we have

$$W = \sigma^{-1}(U) = U' \subseteq X.$$

This proves $Z \subseteq X$.

On the other hand, all such local charts (W, δ, C) with $\text{Spec}(C) = W$ constitute a reduced affine covering \mathcal{C}'_Z of Z with values in Ω that is unique, maximal, and quasi-galois closed over Y by $f \circ \delta$.

In a similar manner, we have $X \subseteq Z$.

Hence, $Z = X$. It is seen that (X, \mathcal{O}_X) and (Z, \mathcal{O}_Z) are essentially equal. \square

Lemma 5.5. (c.f. [9]) *Let $f : X \rightarrow Y$ be a surjective morphism. Then X is qc over Y if there is a unique maximal affine patching \mathcal{C}_X of X with values in Ω satisfying the condition:*

A_α has one and only one conjugate over B_α for any $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ and for any affine open set V_α in Y with $U_\alpha \subseteq f^{-1}(V_\alpha)$, where B_α is the canonical image of $\mathcal{O}_Y(V_\alpha)$ in $k(X)$.

Proof. It is immediate from Lemma 5.4. \square

5.3. Equivalent condition. Now we give an essential and sufficient condition for qc schemes.

Theorem 5.6. *Let $f : X \rightarrow Y$ be a surjective morphism of schemes. The following statements are equivalent:*

- *The scheme X is qc over Y by f .*
- *There is a unique maximal affine covering \mathcal{C}_X of X with values in Ω such that \mathcal{C}_X is quasi-galois closed over Y by f .*
- *There is a unique maximal affine patching \mathcal{C}_X of X with values in Ω such that \mathcal{C}_X is quasi-galois closed over Y by f .*

Proof. It is immediate from Lemmas 5.1, 5.4-5 above and Lemma 5.7 below. \square

We also need the following lemma to prove the third statement in Theorem 5.6.

Lemma 5.7. *Let Z be an integral scheme and let $\mathcal{C}_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}$ be an affine covering of X with values in Ω . Then \mathcal{C}_X induces an affine patching $\mathcal{D}_X = \{(U_\alpha, id_\alpha; \sigma_\alpha(A_\alpha))\}$ of X with values in Ω given in a natural manner:*

$$(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X \mapsto (U_\alpha, id_\alpha; \sigma_\alpha(A_\alpha)) \in \mathcal{D}_X$$

where

$$\sigma_\alpha : A_\alpha \rightarrow A_\alpha$$

is a ring isomorphism induced from the homeomorphism

$$\phi_\alpha^{-1} : \text{Spec}(A_\alpha) \rightarrow U_\alpha.$$

Proof. It is immediate from *Definitions 3.4-5*. \square

6. UNIVERSAL CONSTRUCTION FOR *qc* SCHEMES

In this section we will give a universal construction for a *qc* scheme over a given integral scheme. That is, we will prove the existence of a *qc* cover of an integral scheme.

6.1. A preliminary lemma. Let X be an integral scheme that is not essentially affine in $k(X)$. By the lemma below we can change X into a scheme Z that is essentially affine in $k(Z)$. That is, any integral scheme is isomorphic to an essentially affine scheme.

Lemma 6.1. *For any integral scheme X , there is an integral scheme Z satisfying the properties:*

- $k(X) = k(Z)$;
- $X \cong Z$ are isomorphic schemes;
- Z is essentially affine in the field $k(Z)^{al}$.

Proof. Let $\Omega = k(X)^{al}$ and let ξ be the generic point of X . We have $k(X) \triangleq \mathcal{O}_{X,\xi}$. For any open set U of X , there is the following canonical embedding

$$i_U : \mathcal{O}_X(U) \rightarrow k(X).$$

Now take an affine covering $\mathcal{C}_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of X . Fixed an $\alpha \in \Delta$. We have ring isomorphisms

$$\begin{aligned} \phi_\alpha^\# : A_\alpha &\rightarrow \mathcal{O}_X(U_\alpha); \\ i_{U_\alpha} : \mathcal{O}_X(U_\alpha) &\rightarrow B_\alpha \subseteq k(X). \end{aligned}$$

The isomorphism

$$t_\alpha \triangleq i_{U_\alpha} \circ \phi_\alpha^\# : A_\alpha \rightarrow B_\alpha$$

between rings induces an isomorphism

$$\tau_\alpha : (\text{Spec}(A_\alpha), \mathcal{O}_{\text{Spec}(A_\alpha)}) \rightarrow (\text{Spec}(B_\alpha), \mathcal{O}_{\text{Spec}(B_\alpha)})$$

between schemes.

Then we obtain a new scheme (X, \mathcal{O}'_X) , namely Z , by gluing these schemes $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ along

$$\psi_\alpha \triangleq \tau_\alpha \circ \phi_\alpha$$

for $\alpha \in \Delta$.

It is seen that Z has the desired properties. \square

6.2. A universal construction for qc covers. Fixed a field K (not necessarily of characteristic zero). Let Y be an integral K -variety. Suppose that Y is essentially affine in an algebraic closure of $M \triangleq k(Y)$.

Fixed an extension L of M such that L is Galois over M . Note that here L is not necessarily finitely generated over M .

Lemma 6.2. *There exists an integral K -variety X and a surjective morphism $f : X \rightarrow Y$ such that*

- $L = k(X)$;
- f is affine;
- X is qc over Y by f ;
- X is essentially affine in L^{al} .

Proof. (Universal Construction for qc Covers) Here, we repeat this construction developed in [4]. We will proceed in several steps.

Step 1. Take algebraic closures Ω_M of M and Ω_L of L , respectively. Suppose $\Omega_M \subseteq \Omega_L$.

Let Δ_1 be a transcendental basis of L over M and let Δ_2 be a linear basis of L as a vector space over $M(\Delta_1)$. Put

$$\Delta = \Delta_1 \bigcup \Delta_2.$$

Fixed a reduced affine coverings \mathcal{C}_Y of Y with values in Ω_M from the assumption that Y is essentially affine in Ω_M . Suppose that \mathcal{C}_Y is maximal (by set inclusion).

Step 2. Take a local chart $(V, \psi_V, B_V) \in \mathcal{C}_Y$. It is seen that V is an affine open subset of Y and we have

$$Fr(B_V) = M; \mathcal{O}_Y(V) = B_V \subseteq \Omega_M.$$

Define

$$A_V \triangleq B_V[\Delta_V],$$

i.e., the subring of L generated over B_V by the set

$$\Delta_V \triangleq \{\sigma(w) \in L : \sigma \in Gal(L/M), w \in \Delta\}.$$

Then $Fr(A_V) = L$ holds. It is seen that B_V is exactly the invariant subring of the natural action of the Galois group $Gal(L/M)$ on A_V .

Set

$$i_V : B_V \rightarrow A_V$$

to be the inclusion.

Step 3. Define the disjoint union

$$\Sigma = \coprod_{(V, \psi_V, B_V) \in \mathcal{C}_Y} Spec(A_V).$$

Then Σ is a topological space, where the topology τ_Σ on Σ is naturally determined by the Zariski topologies on all $\text{Spec}(A_V)$.

Let

$$\pi_Y : \Sigma \rightarrow Y$$

be the projection.

Step 4. Given an equivalence relation R_Σ in Σ in such a manner:

For any $x_1, x_2 \in \Sigma$, we say

$$x_1 \sim x_2$$

if and only if

$$j_{x_1} = j_{x_2}$$

holds in L .

Here, j_x denotes the corresponding prime ideal of A_V to a point $x \in \text{Spec}(A_V)$ (see [12]).

Let

$$X = \Sigma / \sim$$

and let

$$\pi_X : \Sigma \rightarrow X$$

be the projection.

It is seen that X is a topological space as a quotient of Σ .

Step 5. Define a map

$$f : X \rightarrow Y$$

by

$$\pi_X(z) \longmapsto \pi_Y(z)$$

for each $z \in \Sigma$.

Step 6. Define

$$\mathcal{C}_X = \{(U_V, \varphi_V, A_V)\}_{(V, \psi_V, B_V) \in \mathcal{C}_Y}$$

where $U_V \triangleq \pi_Y^{-1}(V)$ is an open set in X and $\varphi_V : U_V \rightarrow \text{Spec}(A_V)$ is the identity map on U_V for each $(V, \psi_V, B_V) \in \mathcal{C}_Y$.

Then we have an integral scheme (X, \mathcal{O}_X) by gluing the affine schemes $\text{Spec}(A_V)$ for all local charts $(V, \psi_V, B_V) \in \mathcal{C}_Y$ with respect to the equivalence relation R_Σ (see [12, 15]).

It is seen that \mathcal{C}_X is an affine patching on the scheme X with values in Ω_L .

In particular, \mathcal{C}_X is maximal and quasi-galois closed over Y by f .

By *Theorem 5.6* it is seen that X and f have the desired property. This completes the proof. \square

6.3. Existence of qc covers. Now we give the existence of qc covers.

Theorem 6.3. *Fixed an integral K -variety Y and a Galois extension L over $k(Y)$. Then there exists an integral K -variety X and a surjective morphism $f : X \rightarrow Y$ such that*

- $L = k(X)$;
- f is affine;
- X is a qc over Y by f ;
- X is essentially affine in L^{al} .

Such an integral K -variety X with a morphism f , denoted by (X, f) , is said to be a **qc cover** of Y .

Proof. It is immediate from *Lemmas 6.1-2*. □

Lemma 6.4. *Let X, Y and Z be integral K -varieties such that X and Z are qc over Y . Then X and Z are essentially equal if $k(X) = k(Z)$ and X and Z are isomorphic.*

Proof. It is immediate from *Definition 3.9*. □

Remark 6.5. Let X and Z both be qc over an integral K -variety Y . Suppose $k(X) = k(Z)$. In general, it is not true that X and Z are essentially equal.

7. MAIN PROPERTY OF qc SCHEMES

Let L/K be a field extension. The integral schemes X/Y are said to be a **geometric model** for the extension L/K if there is a group isomorphism $Aut(X/Y) \cong Gal(L/K)$ (e.g., see [13, 18, 19]).

In this section we will prove that qc schemes afford such a geometric model for an extension of a function field.

7.1. Function fields of qc schemes. The function fields of qc schemes are quasi-galois.

Lemma 7.1. *Let X and Y be two integral schemes such that X is qc over Y by a surjective morphism f of finite type. Then the function field $k(X)$ is canonically quasi-galois over the function field $f(Y)$.*

Proof. For brevity, assume that $H \triangleq k(Y)$ is contained in $L \triangleq k(X)$. Let M be a conjugation of L over H .

Fixed an element $w \in M \setminus H$. There is an element $u \in L \setminus H$ and an H -isomorphism $\sigma : L \rightarrow M$ such that $w = \sigma(u)$.

As X is essentially affine in an algebraic closure Ω of L , we must have some affine open set U in X such that u is contained in the ring

$$A \triangleq \mathcal{O}_X(U) \subseteq \Omega$$

where U is contained in $f^{-1}(V)$ for some affine open set V in Y .

Put $B = \sigma(A)$. The ring B is a conjugation of A (canonically) over the ring $\mathcal{O}_Y(V)$. By *Theorem 5.6* it is seen that B must be contained in L and then the element $w \in B$ is contained in L . Hence, we have $M \subseteq L$.

From *Lemma 4.3* it is seen that L is quasi-galois over H . \square

7.2. qc schemes as geometric models. Let X and Y be two integral K -varieties and let $f : X \rightarrow Y$ be a surjective morphism.

Lemma 7.2. *Suppose that X is qc over Y by f and $k(X)$ is canonically Galois over $k(Y)$. Then there is a group isomorphism*

$$\text{Aut}(X/Y) \cong \text{Gal}(k(X)/k(Y)).$$

Proof. In the following we will use the trick developed in [2, 3] to prove the second statement above.

In fact, define a mapping

$$t : \text{Aut}(X/Y) \longrightarrow \text{Gal}(k(X)/k(Y))$$

by

$$\sigma = (\sigma, \sigma^\#) \longmapsto t(\sigma) = \langle \sigma, \sigma^{\#-1} \rangle$$

where $\langle \sigma, \sigma^{\#-1} \rangle$ is the map of $k(X)$ into $k(X)$ given by

$$(U, f) \in \mathcal{O}_X(U) \subseteq k(X) \longmapsto (\sigma(U), \sigma^{\#-1}(f)) \in \mathcal{O}_X(\sigma(U)) \subseteq k(X)$$

for any open set U in X and any element $f \in \mathcal{O}_X(U)$. Here the function field $k(X)$ is taken canonically as the set of elements of the form (U, f) .

It is easily seen that t is well-defined. We will proceed in several steps to prove that t is a group isomorphism.

Step 1. Prove that t is injective. Fixed any $\sigma, \sigma' \in \text{Aut}(X/Y)$ such that $t(\sigma) = t(\sigma')$. We have

$$(\sigma(U), \sigma^{\#-1}(f)) = (\sigma'(U), \sigma'^{\#-1}(f))$$

for any $(U, f) \in k(X)$. In particular, for any $f \in \mathcal{O}_X(U_0)$ we have

$$(\sigma(U_0), \sigma^{\#-1}(f)) = (\sigma'(U_0), \sigma'^{\#-1}(f))$$

where U_0 is an affine open subset of X such that $\sigma(U_0)$ and $\sigma'(U_0)$ are both contained in $\sigma(U) \cap \sigma'(U)$.

It is seen that

$$\sigma|_{U_0} = \sigma'|_{U_0}$$

holds as isomorphisms of schemes. As U_0 is dense in X , we have

$$\sigma = \sigma|_{\overline{U_0}} = \sigma'|_{\overline{U_0}} = \sigma'$$

on the whole of X ; then

$$\sigma(U) = \sigma'(U);$$

hence,

$$\sigma = \sigma'.$$

This proves that t is an injection.

Step 2. Prove that t is surjective. Fixed any element ρ of the group $\text{Gal}(k(X)/k(Y))$.

As $k(X) = \{(U_f, f) : f \in \mathcal{O}_X(U_f) \text{ and } U_f \subseteq X \text{ is open}\}$, we have

$$\rho : (U_f, f) \in k(X) \longmapsto (U_{\rho(f)}, \rho(f)) \in k(X),$$

where U_f and $U_{\rho(f)}$ are open sets in X , f is contained in $\mathcal{O}_X(U_f)$, and $\rho(f)$ is contained in $\mathcal{O}_X(U_{\rho(f)})$.

It is seen that each element of $\text{Gal}(k(X)/k(Y))$ gives a unique element of $\text{Aut}(X/Y)$.

In fact, fixed any affine open set V of Y . It is easily seen that for each affine open set $U \subseteq \phi^{-1}(V)$ there is an affine open set U_ρ in X such that ρ determines an isomorphism λ_U between affine schemes $(U, \mathcal{O}_X|_U)$ and $(U_\rho, \mathcal{O}_X|_{U_\rho})$. Then

$$\lambda_U|_{U \cap U'} = \lambda_{U'}|_{U \cap U'}$$

holds as morphisms of schemes for any affine open sets $U, U' \subseteq \phi^{-1}(V)$.

Glue λ_U along all such affine open subsets $U \subseteq \phi^{-1}(V)$, where V runs through all affine open sets in Y . Then we have an automorphism λ of the scheme X such that $\lambda|_U = \lambda_U$ for any affine open set U in X .

It is clear that $t(\lambda) = \rho$. Hence, t is a surjection.

This completes the proof. \square

Lemma 7.3. *Suppose that X is qc over Y by f and $k(X)$ is canonically Galois over $k(Y)$. Then f is an affine morphism and there is a natural isomorphism*

$$\mathcal{O}_Y \cong f_*(\mathcal{O}_X)^{\text{Aut}(X/Y)}$$

where $(\mathcal{O}_X)^{\text{Aut}(X/Y)}(U)$ denotes the invariant subring of $\mathcal{O}_X(U)$ under the natural action of $\text{Aut}(X/Y)$ for any open subset U of X .

Proof. Let Ω be an algebraic closure of the function field $k(X)$. By Lemma 6.1 assume that Y is essentially affine in Ω without loss of generality. In particular, suppose $k(Y) \subseteq k(X)$ for brevity.

It is clear that X is essentially affine in Ω . Put

$$G = \text{Aut}(X/Y).$$

Fixed a point $x \in X$ and an affine open set U in X with $x \in U$. Then there must be

$$\mathcal{O}_X(U)^G = \mathcal{O}_Y(V)$$

for an affine open set V in Y such that $f(x) \in V$ and $U \subseteq f^{-1}(V)$.

Otherwise, if there is some

$$w \in \mathcal{O}_X(U)^G \setminus \mathcal{O}_Y(V),$$

we have

$$w \notin k(Y), w^{-1} \in k(Y)$$

or

$$w \in k(Y), w^{-1} \notin k(Y).$$

If $w \in k(Y)$ and $w^{-1} \notin k(Y)$, it is seen that $w \notin k(Y)$ holds since we have

$$\sigma(w \cdot w^{-1}) = \sigma(w) \cdot \sigma(w^{-1}) = 1$$

for any $\sigma \in G$.

Hence, for both cases we will have $w \notin k(Y)$, which will be in contradiction to the fact that

$$k(X)^G = k(Y)$$

holds by *Lemma 7.2*.

Now consider any open set U in X . We have

$$\mathcal{O}_X(U)^G = \mathcal{O}_Y(V)$$

for an open set V in Y such that $U \subseteq f^{-1}(V)$ since $\mathcal{O}_X(U)$ can be regarded as a subring of $\mathcal{O}_X(U_0)$ for an affine open set $U_0 \subseteq U$. This prove that

$$\mathcal{O}_Y = (\mathcal{O}_X)^G$$

holds.

Conversely, take any affine open set V of Y . As f is surjective, it is seen that there is an affine open $U \subseteq f^{-1}(V)$ of X such that

$$\mathcal{O}_X(U)^G \supseteq \mathcal{O}_Y(V).$$

Repeating the same procedure above, we can choose U to be such that

$$\mathcal{O}_X(U)^G = \mathcal{O}_Y(V).$$

It follows that

$$U = f^{-1}(V)$$

holds. This proves that f is affine. □

7.3. Main property of qc schemes. The qc schemes behave like quasi-galois extensions of fields.

Theorem 7.4. *Let X and Y be two algebraic K -varieties such that $k(X)$ is separably generated over $k(Y)$ canonically. Suppose that X is qc over Y by a surjective morphism f of finite type. Then there are the following statements:*

- f is affine.
- $k(X)$ is Galois over $k(Y)$ canonically.
- There is a group isomorphism

$$\text{Aut}(X/Y) \cong \text{Gal}(k(X)/k(Y)).$$

In particular, let $\dim X = \dim Y$. Then X is a pseudo-galois cover of Y in the sense of Suslin-Voevodsky.

Proof. It is immediate from *Theorem 4.1* and *Lemmas 7.1-3*. □

Here for pseudo-galois cover, see [18, 19] for definition and property.

8. sp -COMPLETION

By the graph functor, there is an sp -completion of a given integral scheme, which is an integral scheme of the same length but have the maximal combinatorial graph.

The sp -completion can give a type of completions of rational maps between schemes.

In the present paper, the sp -completion will be applied to definitions of formally unramified extension of fields in §9 and of monodromy actions of automorphism groups in §12.

8.1. The graph functor Γ from schemes to graphs. For convenience, in this subsection we will review the graph functor developed in [1, 9]. See [1, 9] for proofs of the results listed below.

Let X be a scheme. Take any points x, y in X .

If y is in the (topological) closure $\overline{\{x\}}$, y is a **specialization** of x (or, x is a **generalization** of y) in X , denoted by $x \rightarrow y$.

Put $Sp(x) = \{y \in X \mid x \rightarrow y\}$. Then $Sp(x) = \overline{\{x\}}$ is an irreducible closed subset in X .

If $x \rightarrow y$ and $y \rightarrow x$ both hold in X , y is a **generic specialization** of x in X , denoted by $x \leftrightarrow y$.

x is said to be **generic** (or **initial**) in X if we must have $x \leftrightarrow z$ for any $z \in X$ such that $z \rightarrow x$.

x is said to be **closed** (or **final**) if we must have $x \leftrightarrow z$ for any $z \in X$ such that $x \rightarrow z$.

We have the following preliminary facts for specializations.

Lemma 8.1. ([1, 9]) *For any points $x, y \in X$, we have $x \leftrightarrow y$ in X if and only if $x = y$.*

Lemma 8.2. ([1, 9]) *Fixed any specialization $x \rightarrow y$ in X . Then there is an affine open subset U of X such that the two points x and y are both contained in U . In particular, any affine open set in X containing (the specialization) y must contain (the generalization) x .*

Lemma 8.3. ([1, 9]) *Any morphism between schemes is specialization-preserving. That is, fixed any morphism $f : X \rightarrow Y$ between schemes. Then there is a specialization $f(x) \rightarrow f(y)$ in Y for any specialization $x \rightarrow y$ in X .*

Let's recall preliminary definitions on combinatorial graphs (see [20]). Fixed a graph G . Let $V(G)$ be the **set of vertices** in G and $E(G)$ the **set of edges** in G .

A graph H is a **subgraph** of G , denoted by $H \subseteq G$, provided that $V(G) \supseteq V(H)$, $E(G) \supseteq E(H)$, and every $L \in E(H)$ has the same ends in H as in G .

Now we obtain the graph functor Γ from *Lemmas 8.1-3*.

Lemma 8.4. (Graph Functor [1, 9]) *There exists a covariant functor Γ , called the **graph functor**, from the category Sch of schemes to the category $Grph$ of combinatorial graphs, given in such a natural manner:*

- *For a scheme X , $\Gamma(X)$ is the graph in which the vertex set is the set of points in the underlying space X and the edge set is the set of specializations in X . Here, for any points $x, y \in X$, we say that there is an edge from x to y if and only if there is a specialization $x \rightarrow y$ in X .*
- *For a morphism $f : X \rightarrow Y$ of schemes, $\Gamma(f) : \Gamma(X) \rightarrow \Gamma(Y)$ is the homomorphism between graphs. Here, any specialization $x \rightarrow y$ in the scheme X as an edge in $\Gamma(X)$, is mapped by $\Gamma(f)$ into the specialization $f(x) \rightarrow f(y)$ as an edge in $\Gamma(Y)$.*

8.2. *sp*-completion. Let G and H be combinatorial graphs. Recall that an **isomorphism** t from G onto H is a ordered pair (t_V, t_E) satisfying the conditions:

- t_V is a bijection from $V(G)$ onto $V(H)$;
- t_E is a bijection from $E(G)$ onto $E(H)$;
- Let $x \in V(G)$ and $L \in E(G)$. Then x is incident with L if and only if $t_V(x) \in V(H)$ is incident with $t_E(L) \in E(H)$.

The following definition says that the graph of a given integral scheme is maximal relative to its function field.

Definition 8.5. An integral scheme X is said to be *sp-complete* if X must be essentially equal to Y for any integral scheme Y such that

- $\Gamma(X)$ is isomorphic to a subgraph of $\Gamma(Y)$;
- $k(Y)$ is contained in a separable closure of $k(X)$.

Remark 8.6. The function field of an *sp-complete* integral scheme must be a separable closure. In other words, there is no other separably closed points that can be added to an *sp-complete* integral scheme.

Now we give the existence of the *sp-completion* of an integral scheme.

Theorem 8.7. *For any integral scheme X , there is an integral scheme X_{sp} and a surjective morphism $\lambda_X : X_{sp} \rightarrow X$ satisfying the following properties:*

- λ_X is affine;
- X_{sp} is *sp-complete*;
- X_{sp} is essentially affine in $k(X)^{al}$;
- $k(X_{sp})$ is a separable closure of $k(X)$;
- X_{sp} is quasi-galois closed over X by λ_X .

Such a scheme X_{sp} with a morphism λ_X , denoted by (X_{sp}, λ_X) , is said to be an *sp-completion* of X . We will denote by $Sp[X]$ the set of all *sp-completions* of an integral scheme X .

Proof. (**Universal Construction for *sp-Completion***) Here repeat the construction developed in [9].

Let $K = k(X)$ and $L = K^{sep}$. Fixed a transcendental basis Δ_1 of L over K and a linear basis Δ_2 of L as a vector space over $K(\Delta_1)$. Put

$$G = Gal(L/K); \Delta = \Delta_1 \bigcup \Delta_2.$$

By *Lemma 6.1*, without loss of generality, assume that X has a reduced affine covering \mathcal{C}_X with values in L . We choose \mathcal{C}_X to be maximal (by set inclusion).

We proceed in several steps such as the following to give the construction:

- Fixed a local chart $(V, \psi_V, B_V) \in \mathcal{C}_X$. Define $A_V = B_V[\Delta_V]$, where $\Delta_V = \{\sigma(x) \in L : \sigma \in G, x \in \Delta\}$. Set $i_V : B_V \rightarrow A_V$ to be the inclusion.
- Let

$$\Sigma = \coprod_{(V, \psi_V, B_V) \in \mathcal{C}_X} Spec(A_V)$$

be the disjoint union. Denote by $\pi_X : \Sigma \rightarrow X$ the projection induced by the inclusions i_V .

- Given an equivalence relation R_Σ in Σ in such a manner:

For any $x_1, x_2 \in \Sigma$, we say $x_1 \sim x_2$ if and only if $j_{x_1} = j_{x_2}$ holds in L . Here j_x denotes the corresponding prime ideal of A_V to a point $x \in \text{Spec}(A_V)$.

Let X_{sp} be the quotient space Σ / \sim and let $\pi_{sp} : \Sigma \rightarrow X_{sp}$ be the projection of spaces.

- Set a map $\lambda_X : X_{sp} \rightarrow X$ of topological spaces by $\pi_{sp}(z) \mapsto \pi_X(z)$ for each $z \in \Sigma$.
- Suppose

$$\mathcal{C}_{X_{sp}} = \{(U_V, \varphi_V, A_V)\}_{(V, \psi_V, B_V) \in \mathcal{C}_X}.$$

Here $U_V = \pi_X^{-1}(V)$ and $\varphi_V : U_V \rightarrow \text{Spec}(A_V)$ is the identity map for each $(V, \psi_V, B_V) \in \mathcal{C}_X$.

- There is a scheme, namely X_{sp} , by gluing the affine schemes $\text{Spec}(A_V)$ for all $(U_V, \varphi_V, A_V) \in \mathcal{C}_X$ with respect to the equivalence relation R_Σ . Naturally, λ_X becomes a morphism of schemes.

It is seen that X_{sp} and λ_X are the desired scheme and morphism, respectively. \square

There is the uniqueness of sp -completions such as the following.

Lemma 8.8. *Fixed any integral K -varieties X . Then all sp -completions of X are essentially equal.*

Proof. It is seen from Definition 8.5, Remark 8.6, and Theorem 8.7. \square

Lemma 8.9. *Fixed any two integral K -varieties X and Y . Suppose that $k(X)$ and $k(Y)$ have the same separable closure. Then either*

$$Sp[X] = Sp[Y]$$

or

$$Sp[X] \cap Sp[Y] = \emptyset$$

holds.

Proof. It is immediate from Lemma 8.8. \square

Remark 8.10. An sp -completion of an integral scheme is sp -complete. By sp -completion we can give a completion of rational maps between integral schemes.

Remark 8.11. An integral scheme X and its sp -completion X_{sp} have the same dimension. However, the sp -completion is very complicated and exotic. In general, it is not true that X_{sp} is of finite type over X . For example, let t be a variable over \mathbb{Q} . It is seen that $\text{Spec}(\overline{\mathbb{Q}})$ and

$\text{Spec}(\overline{\mathbb{Q}(t)})$ are sp -completions of $\text{Spec}(\mathbb{Q})$ and $\text{Spec}(\mathbb{Q}(t))$, respectively. Their underlying spaces are very different.

Remark 8.12. By *Theorem 8.7* it is seen that an sp -completion of an integral scheme behaves like a separable closure of a field.

9. UNRAMIFIED EXTENSIONS OF FUNCTION FIELDS

In this section we will use sp -complete schemes to introduce a notion of formally unramified extensions over function fields and then give several preliminary properties. The formally unramified extensions will be applied to the computation of étale fundamental groups.

9.1. Basic lemma. Fixed a field K . We have the following basic result.

Lemma 9.1. *Let X and Y be two integral K -varieties satisfying the two conditions:*

- $Sp[X] = Sp[Y]$ are equal sets.
- $Aut(X_{sp}/X) \cong Aut(Y_{sp}/Y)$ are isomorphic groups.

Then X and Y are isomorphic schemes.

Proof. Fixed any sp -completions (X_{sp}, λ_X) of X and (Y_{sp}, λ_Y) of Y , respectively. As $Sp[X] = Sp[Y]$, we have an isomorphism

$$t : X_{sp} \rightarrow Y_{sp}.$$

Let

$$\sigma : Aut(X_{sp}/X) \rightarrow Aut(Y_{sp}/Y)$$

be an isomorphism between groups.

Take any point $x_0 \in X$. From *Lemmas 7.2-3* we have

$$\lambda_X^{-1}(x_0) = \{g(x_0) \in X_{sp} : g \in Aut(X_{sp}/X)\};$$

$$\lambda_Y^{-1}(t(x_0)) = \{h(t(x_0)) \in Y_{sp} : h \in Aut(Y_{sp}/Y)\}.$$

It follows that there exists a morphism $f_{sp} : X_{sp} \rightarrow Y_{sp}$ given by

$$g(x_0) \mapsto \sigma(g)(t(x_0)).$$

From f_{sp} we obtain a morphism $f : X \rightarrow Y$ given by

$$x_0 = \lambda_X(g(x_0)) \mapsto \lambda_Y(\sigma(g)(t(x_0)))$$

satisfying the property

$$f \circ \lambda_X = \lambda_Y \circ f_{sp}.$$

It is easily seen that $f_{sp} : X_{sp} \rightarrow Y_{sp}$ is an isomorphism. Hence, $f : X \rightarrow Y$ is an isomorphism. \square

9.2. Formally unramified extensions. Fixed an integral K -variety X over a field K . Let L_1 and L_2 be two algebraic extensions over the function field $k(X)$, respectively.

Definition 9.2. L_2 is said to be a **finite X -formally unramified Galois extension** over L_1 if there are two integral K -varieties X_1 and X_2 and a surjective morphism $f : X_2 \rightarrow X_1$ such that

- $Sp[X] = Sp[X_1] = Sp[X_2]$;
- $k(X_1) = L_1$, $k(X_2) = L_2$;
- X_2 is a finite étale Galois cover of X_1 by f .

In such a case, X_2/X_1 are said to be a **X -geometric model** of the field extension L_2/L_1 .

Remark 9.3. It is seen that such geometric models are unique up to isomorphisms. It follows that the formally unramified extension above is well-defined. In fact, fixed any two geometric models X_2/X_1 and Y_2/Y_1 for the extension L_2/L_1 , respectively. By *Lemma 9.1* we must have isomorphisms $X_2 \cong Y_2$ and $X_1 \cong Y_1$, respectively. This is due to the preliminary fact that we have

$$Aut(X_{sp}/X_i) \cong Gal(k(X)^{sep}/L_i) \cong Aut(X_{sp}/Y_i)$$

for $i = 1, 2$.

Remark 9.4. Suppose that L_3/L_2 and L_2/L_1 both are X -formally unramified extensions. Then L_3/L_1 must be X -formally unramified.

Remark 9.5. Note that even for the case that L_1 and L_2 are both algebraic extensions of K , in general, the formally unramified defined in *Definition 9.2* does not coincide with unramified that is defined in algebraic number theory.

Remark 9.6. Note that we define another unramified extensions in [5, 6, 9] for arithmetic schemes, which is a generalization of unramified extensions in algebraic number theory and hence is different from the above one defined in *Definition 9.2*.

Definition 9.7. Let X be an integral K -variety over a field K . Set

$$k(X)^{au} \triangleq \text{the smallest field containing all finite } X\text{-formally unramified subextensions over } L \text{ contained in } L^{al}.$$

The field $k(X)^{au}$ is said to be the **maximal formally unramified extension** of the function field $k(X)$.

Lemma 9.8. *Let X be an integral K -variety and let $L \subseteq k(X)^{au}$ be a finite Galois extension of $k(X)$. Then there are the following statements.*

- (i) $k(X)^{au}$ is an algebraic Galois extension of $k(X)$. In particular, $k(X)^{au}$ is a subfield of $k(X)^{sep}$.
- (ii) There is a finite X -formally unramified Galois extension M of $k(X)$ such that $M \supseteq L$.
- (iii) Let M be a finite X -formally unramified Galois extension of $k(X)$ such that $M \supseteq L$. Then so is M over L .
- (iv) L is a finite X -formally unramified Galois extension of $k(X)$.

Proof. (i) It is immediate from preliminary facts on field theory.

(ii) It is clear from the assumption that $L \subseteq k(K)^{au}$ holds.

(iii) Take a geometric model X_M/X for the extension $M/k(X)$. It reduces to the case that X_M and X are both affine schemes.

Suppose

$$X = \text{Spec}(K_0), K_0 = K[t_1, t_2, \dots, t_n];$$

$$L = \text{Fr}(L_0), L_0 = K[t_1, t_2, \dots, t_n, s_1, s_2, \dots, s_l];$$

$$X_M = \text{Spec}(M_0), M_0 = K[t_1, t_2, \dots, t_n, s_1, s_2, \dots, s_l, s_{l+1}, \dots, s_m].$$

Here, the elements

$$t_1, t_2, \dots, t_n, s_1, s_2, \dots, s_l, s_{l+1}, \dots, s_m$$

are all contained in $k(X)^{au}$, and

$$s_1, s_2, \dots, s_l$$

and

$$s_1, s_2, \dots, s_l, s_{l+1}, \dots, s_m$$

are both supposed to contain all conjugates over the field $k(X)$.

It is easily seen that X_M is a finite étale Galois cover of $\text{Spec}(L_0)$ from base change of étale morphisms.

(iv) It reduces to consider affine schemes. Take K_0, L_0, M_0 as in (iii) above.

Let \mathfrak{P} be a maximal ideal of M_0 . Just check what one has done in algebraic number theory. Put

$$\mathfrak{p} = \mathfrak{P} \cap K_0.$$

Then \mathfrak{p} is a maximal ideal of K_0 .

Conversely, let \mathfrak{p} be a maximal ideal of K_0 . From definition for étale morphisms, we have one and only one maximal ideal \mathfrak{P} of M_0 that is over the maximal ideal \mathfrak{p} .

Likewise, there is one and only one maximal ideal \mathfrak{P}_0 of L_0 such that $\mathfrak{P}|\mathfrak{P}_0$ and $\mathfrak{P}_0|\mathfrak{p}$ hold.

It follows that $\text{Spec}(L_0)$ must be unramified over X and hence étale over X . This completes the proof. \square

9.3. Arithmetic unramified extension. There is another type of unramified extensions, the arithmetic unramified extensions over the ring \mathcal{O}_K of algebraic integers of a number field K .

Convention. In this subsection, an **integral \mathbb{Z} -variety** is defined to be an integral scheme surjectively over $\text{Spec}(\mathbb{Z})$; an **arithmetic variety** is an integral scheme surjectively over $\text{Spec}(\mathbb{Z})$ of finite type.

Likewise, we have the following basic lemma.

Lemma 9.9. *Let X and Y be two integral \mathbb{Z} -varieties satisfying the two conditions:*

- $Sp[X] = Sp[Y]$ are equal sets.
- $Aut(X_{sp}/X) \cong Aut(Y_{sp}/Y)$ are isomorphic groups.

Then X and Y are isomorphic schemes.

Fixed an integral \mathbb{Z} -variety X over a field K . Let L_1 and L_2 be two algebraic extensions over the function field $k(X)$, respectively.

Definition 9.10. The field L_2 is said to be a **finite X -unramified Galois extension** over L_1 if there are two integral \mathbb{Z} -varieties X_1 and X_2 and a surjective morphism $f : X_2 \rightarrow X_1$ such that

- $Sp[X] = Sp[X_1] = Sp[X_2]$;
- $k(X_1) = L_1, k(X_2) = L_2$;
- X_2 is a finite étale Galois cover of X_1 by f .

In such a case, X_2/X_1 are said to be a **X -geometric model** of the field extension L_2/L_1 .

Remark 9.11. Let $L_1 \subseteq L_2 \subseteq L_3$ be function fields over a number field K . Suppose that L_2/L_1 and L_3/L_2 are X -unramified extensions. Then L_3 is X -unramified over L_1 .

Definition 9.12. Let X be an integral \mathbb{Z} -variety. Set

$k(X)^{un} \triangleq$ the smallest field containing all finite X -unramified subextensions over L contained in L^{al} .

The field $k(X)^{un}$ is said to be the **maximal unramified extension** of the function field $k(X)$.

Lemma 9.13. *Let X be an integral \mathbb{Z} -variety and let $L \subseteq k(X)^{un}$ be a finite Galois extension of $k(X)$. Then there are the following statements.*

- (i) $k(X)^{un}$ is an algebraic Galois extension of $k(X)$.
- (ii) There is a finite X -unramified Galois extension M of $k(X)$ such that $M \supseteq L$.
- (iii) Let M be a finite X -unramified Galois extension of $k(X)$ such that $M \supseteq L$. Then so is M over L .

(iv) L is a finite X -unramified Galois extension of $k(X)$.

Proof. Repeat what we have done in proving Lemma 9.8. \square

Remark 9.14. It is seen that for the case of an algebraic extension, the unramified extension defined in Definition 9.12 coincides exactly with that in algebraic number theory.

It appears that unramified extensions for an arithmetic variety and for an algebraic K -variety have some common properties. However, they are very different. For example, we have

$$k(\operatorname{Spec}(\mathbb{Z})) = \mathbb{Q}^{un}; \quad k(\operatorname{Spec}(\mathbb{Q})) = \mathbb{Q}^{sep} = \overline{\mathbb{Q}}.$$

In particular, for arithmetic varieties, we have a stronger result such as the following.

Theorem 9.15. *Let X and Y be two arithmetic varieties such that $k(X) = k(Y)$. Then $Sp[X] = Sp[Y]$ holds, i.e., X and Y have the same sp -completions; moreover, X and Y are isomorphic.*

Proof. By Lemma 9.9 it suffices to prove that X and Y have a common sp -completion. Fixed any sp -completions (X_{sp}, λ_X) of X and (Y_{sp}, λ_Y) of Y , respectively. It reduces to prove that X_{sp} and Y_{sp} are isomorphic.

In the following we will proceed in several steps to prove that there exists an isomorphism

$$f_{sp} : X_{sp} \rightarrow Y_{sp}.$$

Step 1. We have $\Omega \triangleq k(X)^{al} = k(Y)^{al}$. As X_{sp} and Y_{sp} are both qc over $\operatorname{Spec}(\mathbb{Z})$, it is seen that for any affine open set U in X there must be an affine open set V in Y such that

$$f_U : U \rightarrow V$$

is an isomorphism of schemes which is induced from an isomorphism

$$\sigma_U : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(U)$$

between subrings of Ω .

By the universal construction in §8.2, we have a morphism

$$f_{sp} : X_{sp} \rightarrow Y_{sp}$$

such that there is the restriction

$$f_{sp}|_U = f_U$$

to each affine open set U in X .

Step 2. By Theorems 5.6, 8.7, assume that σ_U is an identity map without loss of generality. It is seen that f_{sp} is an injective morphism.

Step 3. Take any closed point y_0 in Y_{sp} . Let $B(y_0) \subseteq \Omega$ be a subring such that the affine open set $V(y_0) = \text{Spec}(B(y_0))$ in Y containing the point y_0 . Denote by j_{y_0} the prime ideal in $B(y_0)$ corresponding to y_0 .

By *Theorem 5.6* it is seen that $B(y_0)$ is a ring over \mathbb{Z} generated by the set

$$\{t_1, \dots, t_n\} \cup \Delta$$

where t_1, \dots, t_n are variables over \mathbb{Q} , $\dim X = n$, and

$$\Delta = \overline{\mathbb{Q}(t_1, \dots, t_n)} \setminus \mathbb{Z}.$$

This is due to the fact that Y_{sp} is *qc* over $\text{Spec}(\mathbb{Z})$.

It is seen that such a prime ideal j_{y_0} contains a unique prime $\mathfrak{P} \in \mathcal{O}_K$ over a prime $\mathfrak{p} \in \mathbb{N}$, where \mathcal{O}_K is the ring of the algebraic integers of a number field K and j_{y_0} is a maximal ideal of the ring $B(y_0)$ generated by a set $\Delta_{\mathfrak{P}}$ containing the subset

$$\{\mathfrak{P}\} \cup \{t_1, \dots, t_n\}$$

of Δ .

As X_{sp} is *qc* over $\text{Spec}(\mathbb{Z})$, we have a point $x_0 \in X$ such that

$$j_{x_0} = j_{y_0}.$$

Then we have $f_{sp}(x_0) = y_0$.

This completes the proof. \square

Remark 9.16. By *Theorem 9.15* it is seen that *unramified extension*, the notion for arithmetic varieties given in [5, 6, 9], as in *Definition 9.12*, are well-defined.

10. ALGEBRAIC FUNDAMENTAL GROUPS

In this section we will give the computation of algebraic fundamental groups.

10.1. A universal cover for an étale fundamental group. For an integral K -variety X , let $k(X)^{au}$ denote the maximal formally unramified extension of the function field $k(X)$.

Lemma 10.1. *For any integral K -variety X , there exists an integral K -variety X_{et} and a surjective morphism $p_X : X_{et} \rightarrow X$ satisfying the properties:*

- p_X is affine;
- $k(X_{et}) = k(X)^{au}$;
- X_{et} is *qc* over X by p_X ;
- $k(X_{et})$ is Galois over $k(X)$;
- X_{et} is essentially affine in $k(X)^{au}$.

Such an integral K -variety X_{et} with a morphism p_X , denoted by (X_{et}, p_X) , is called a **universal cover** over X for the étale fundamental group $\pi_1^{et}(X)$.

Proof. (Universal Construction for the Cover) By *Lemma 6.1*, without loss of generality, assume that X has a reduced affine covering \mathcal{C}_X with values in $k(X)^{al}$. Let \mathcal{C}_X be maximal by set inclusion.

We will proceed in several steps:

- Fixed a set Δ of generators of the field $k(X)^{au}$ over $k(X)$.
- For any local chart $(V, \psi_V, B_V) \in \mathcal{C}_X$, define $A_V = B_V[\Delta_V]$, that is, A_V over B_V generated by the set

$$\Delta_V = \{\sigma(x) \in k(X)^{au} : \sigma \in \text{Gal}(k(X)^{au}/k(X)), x \in \Delta\}.$$

Let $i_V : B_V \rightarrow A_V$ be the inclusion.

- Assume that

$$\Sigma = \coprod_{(V, \psi_V, B_V) \in \mathcal{C}_X} \text{Spec}(A_V)$$

is the disjoint union. Let $\pi_X : \Sigma \rightarrow X$ be the projection induced by the inclusions i_V .

- Define an equivalence relation R_Σ in Σ in such a manner:

For any $x_1, x_2 \in \Sigma$, we say $x_1 \sim x_2$ if and only if $j_{x_1} = j_{x_2}$ holds in L , where j_x denotes the corresponding prime ideal of A_V to a point x in $\text{Spec}(A_V)$.

Let X_{et} be the quotient space Σ / \sim and let $\pi_{et} : \Sigma \rightarrow X_{et}$ be the projection of spaces.

- Set a map $p_X : X_{et} \rightarrow X$ of spaces by $\pi_{et}(z) \mapsto \pi_X(z)$ for each $z \in \Sigma$.
- Suppose

$$\mathcal{C}_{X_{et}} = \{(U_V, \varphi_V, A_V)\}_{(V, \psi_V, B_V) \in \mathcal{C}_X}.$$

Here $U_V = \pi_X^{-1}(V)$ and $\varphi_V : U_V \rightarrow \text{Spec}(A_V)$ is the identity map for each $(V, \psi_V, B_V) \in \mathcal{C}_X$.

- There is a scheme, namely X_{et} , obtained by gluing the affine schemes $\text{Spec}(A_V)$ for all $(U_V, \varphi_V, A_V) \in \mathcal{C}_X$ with respect to the equivalence relation R_Σ . Naturally, p_X becomes a morphism of schemes.

It is seen that X_{et} and p_X satisfy the properties. This completes the proof. \square

10.2. A computation of étale fundamental groups. By *Lemma 10.1* we have the following result.

Theorem 10.2. *For any integral K -variety X , there exists a group isomorphism*

$$\pi_1^{et}(X) \cong \text{Gal}(k(X)^{au}/k(X)).$$

Proof. Assume that X has a reduced affine covering \mathcal{C}_X with values in $k(X)^{al}$ without loss of generality.

Let $\Delta \subseteq k(X)^{au} \setminus k(X)$ be a set of generators of the field $k(X)^{au}$ over $k(X)$. Put

$$I = \{\text{finite subsets of } \Delta\}.$$

We will proceed in several steps to give the proof.

Step 1. Fixed any α in I . Repeating the universal construction in §10.1 for α , i.e., replacing Δ by α , we have an integral K -variety X_α and a surjective morphism $f_\alpha : X_\alpha \rightarrow X$ satisfying the properties:

- f_α is affine;
- $k(X_\alpha) \subseteq k(X)^{au}$;
- X_α is *qc* over X by f_α ;
- $k(X_\alpha)$ is Galois over $k(X)$;
- X_α is essentially affine in $k(X)^{au}$.

Step 2. Let $\alpha \subseteq \beta$ be in I . By *Step 1* we have integral K -varieties X_α and X_β which are *qc* over X , respectively. There is a surjective morphism $f_\alpha^\beta : X_\beta \rightarrow X_\alpha$ satisfying the properties:

- f_α^β is affine;
- $f_\beta = f_\alpha \circ f_\alpha^\beta$;
- X_β is *qc* over X_α by f_α^β ;
- $k(X_\beta)$ is Galois over $k(X_\alpha)$.

Here f_α^β is obtained in a canonical manner similar to f_α .

It is clear that there is a γ in I such that $\gamma \supseteq \alpha$ and $\gamma \supseteq \beta$. Hence, we have an integral K -variety X_γ that is *qc* over X_β and over X_α , respectively.

Step 3. For any α, β in I , we say $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$. Then I is a partially ordered set.

Hence, $\{k(X_\alpha); i_\alpha^\beta\}_{\alpha \in I}$ is a direct system of groups, where each

$$i_\alpha^\beta : k(X_\alpha) \rightarrow k(X_\beta)$$

is a homomorphism of fields canonically induced by f_α^β .

Let (X_{et}, p_X) be a universal cover for $\pi_1^{et}(X)$. For the fields, we have

$$k(X_{et}) = k(X)^{au} = \lim_{\longrightarrow \alpha \in I} k(X_\alpha).$$

For the Galois groups, we have

$$\mathrm{Gal}(k(X_{et})/k(X)) \cong \varprojlim_{\alpha \in I} \mathrm{Gal}(k(X_\alpha)/k(X)).$$

Step 4. Let

$$[X]_{au} = \{X_\alpha : \alpha \in I\}.$$

Then $[X]_{au}$ is a directed set. Here for any $X_\alpha, X_\beta \in [X]_{au}$, we say

$$X_\alpha \leq X_\beta$$

if and only if X_β is *qc* over X_α .

Fixed a geometric point s of X over $k(X)^{al}$. Put

$$[X]_{et} = \{\text{finite étale Galois covers of } X \text{ over } s\}.$$

Then $[X]_{et}$ is a directed set. Here for any $X_1, X_2 \in [X]_{et}$, we say

$$X_1 \leq X_2$$

if and only if X_2 is a finite étale Galois cover over X_1 .

Step 5. Fixed any $X_\alpha, X_\beta \in [X]_{au}$.

It is seen that X_α and X_β both are finite étale Galois covers of X by *Lemma 9.8*.

Let X_β be *qc* over X_α . Then X_β is a étale finite Galois cover of X_α from *Lemma 9.8* again.

Hence, $[X]_{au}$ is a directed subset of $[X]_{et}$.

Step 6. Let $Z \in [X]_{et}$. We have $k(Z) \subseteq k(X)^{au}$. It is seen that $k(Z)$ is a finite unramified Galois extension of $k(X)$.

Let $\alpha \subseteq k(Z) \setminus k(X)$ be a set of generators of the field $k(Z)$ over $k(X)$. As $\alpha \in I$ is finite and Δ is infinite, there is a finite set $\beta \in I$ such that

$$\alpha \subsetneq \beta \subsetneq \Delta.$$

We have

$$X_\beta \in [X]_{au}$$

such that X_β is *qc* over Z .

Hence, $[X]_{au}$ is a co-final directed subset in X_{et} .

Step 7. Now by *Steps 1-6* above we have

$$\begin{aligned} \pi_1^{et}(X) &= \lim_{\leftarrow Z \in [X]_{et}} \mathrm{Aut}(Z/X) \\ &\cong \lim_{\leftarrow Z \in [X]_{au}} \mathrm{Aut}(Z/X) \\ &\cong \lim_{\leftarrow Z \in [X]_{au}} \mathrm{Gal}(k(Z)/k(X)) \\ &= \lim_{\leftarrow \alpha \in I} \mathrm{Gal}(k(X_\alpha)/k(X)) \\ &\cong \mathrm{Gal}(k(X_{et})/k(X)) \\ &= \mathrm{Gal}(k(X)^{au}/k(X)). \end{aligned}$$

This completes the proof. \square

Remark 10.3. Let X be an integral K -variety. From *Theorem 10.2* we have

$$\pi_1^{et}(X_{et}) \cong \{0\}.$$

For example, let $X = \text{Spec}(\mathbb{Q})$. We have $X_{et} = \text{Spec}(\overline{\mathbb{Q}})$ and hence

$$\pi_1^{et}(\text{Spec}(\overline{\mathbb{Q}})) \cong \{0\}.$$

10.3. A prior estimate of the étale fundamental group. In this subsection we will introduce the qc fundamental group of an algebraic variety. We will prove that the étale fundamental group is a normal subgroup of the qc fundamental group.

Fixed an algebraic K -variety X . Let Ω be the separable closure of a separably generated extension of the function field $k(X)$.

Define

$$[X; \Omega]_{qc}$$

to be the set of algebraic K -varieties Z satisfying the conditions:

- $k(Z)$ is contained in Ω ;
- There is a surjective morphism $f : Z \rightarrow X$ of finite type such that Z is qc over X .

In [7], we require such an additional condition that

Z has a reduced affine covering with values in Ω .

However, from *Lemma 6.1* it is seen that there is no essential difference between the two conditions.

There are preliminary facts on the set $[X; \Omega]_{qc}$ such as the following.

Lemma 10.4. *For any $Z_1, Z_2 \in [X; \Omega]_{qc}$, there is a third $Z_3 \in [X; \Omega]_{qc}$ such that Z_3 is qc over Z_1 and Z_2 , respectively.*

Lemma 10.5. *Let $Z_1, Z_2, Z_3 \in [X; \Omega]_{qc}$. Suppose that Z_2 is qc over Z_1 and Z_3 is qc over Z_2 . Then Z_3 is qc over Z_1 .*

Here, *Lemmas 10.4-5* above can be proved in a manner similar to what we have done for the proof of *Theorem 10.2*.

Set a partial order \leq in the set $[X; \Omega]_{qc}$ in such a manner:

For any $Z_1, Z_2 \in [X; \Omega]_{qc}$, we say

$$Z_1 \leq Z_2$$

if and only if there is a surjective morphism $\varphi : Z_2 \rightarrow Z_1$

of finite type such that Z_2 is qc over Z_1 .

By *Lemmas 10.4-5* it is seen that $[X; \Omega]_{qc}$ is a directed set and

$$\{\text{Aut}(Z/X) : Z \in [X; \Omega]_{qc}\}$$

is an inverse system of groups.

Now we introduce the following definition.

Definition 10.6. Let X be an algebraic K -variety. Suppose that Ω is the separable closure of a separably generated extension of $k(X)$. The inverse limit

$$\pi_1^{qc}(X; \Omega) \triangleq \varprojlim_{Z \in [X; \Omega]_{qc}} \text{Aut}(Z/X)$$

of the inverse system $\{\text{Aut}(Z/X) : Z \in [X; \Omega]_{qc}\}$ of groups is said to be the **qc fundamental group** of X with coefficients in Ω .

We have the following result on the qc fundamental group, which will be prove in §10.5.

Theorem 10.7. *Let X be an algebraic K -variety. Suppose that Ω is the separable closure of a separably generated extension of $k(X)$. There are the following statements.*

(i) *There is a group isomorphism*

$$\pi_1^{qc}(X; \Omega) \cong \text{Gal}(\Omega/k(X)).$$

(ii) *There is a group isomorphism*

$$\pi_1^{et}(X; s) \cong \pi_1^{qc}(X; \Omega)_{et}$$

for a geometric point s of X over Ω , where $\pi_1^{qc}(X; \Omega)_{et}$ is a normal subgroup of $\pi_1^{qc}(X; \Omega)$.

Remark 10.8. Let X be an algebraic K -variety. Define

$$\pi_1^{qc}(X) \triangleq \pi_1^{qc}(X; k(X)^{sep}).$$

It is seen that there is a group isomorphism

$$\pi_1^{qc}(X) \cong \text{Gal}(k(X)^{sep}/k(X)).$$

Remark 10.9. Let X be an algebraic K -variety. The quotient group

$$\pi_1^{br}(X) = \frac{\pi_1^{qc}(X; k(X)^{sep})}{\pi_1^{qc}(X; k(X)^{sep})_{et}}$$

is said to be the **ramified group** (or **branched group**) of X . By *Lemma 9.8* it is seen that $k(X)^{sep}$ is a Galois extension of $k(X)^{au}$; from *Theorems 10.2, 10.7* we have

$$\pi_1^{br}(X) = \frac{\text{Gal}(k(X)^{sep}/k(X))}{\text{Gal}(k(X)^{au}/k(X))}.$$

The ramified group $\pi_1^{br}(X)$ reflects the topological properties of the scheme X such as the branched covers. In deed, such ramified groups will play an important role in giving the anabelian functors in the present paper.

10.4. A universal cover for the qc fundamental group. Let X be an algebraic K -variety. Suppose that Ω is the separable closure of a separably generated extension of the function field $k(X)$.

Assume that X has a reduced affine covering with values in Ω^{al} without loss of generality. Let

$$G = Gal(\Omega/k(X))$$

and let

$$\Delta \subseteq \Omega \setminus k(X)$$

be a set of generators of Ω over $k(X)$. By *Theorem 4.1* it is seen that Ω is Galois over $k(X)$.

Repeating the universal construction for étale fundamental group in §10.1, we have an integral variety X_Ω and a morphism f_Ω such as in the following lemma.

Lemma 10.10. *There is an integral K -variety X_Ω and a surjective morphism $f_\Omega : X_\Omega \rightarrow X$ satisfying the conditions:*

- $k(X_\Omega) = \Omega$;
- f_Ω is affine;
- X_Ω is qc over X by f_Ω ;
- $k(X_\Omega)$ is Galois over $k(X)$;
- X_Ω is essentially affine in Ω .

Such an integral K -variety X_Ω with a morphism f_Ω , denoted by (X_Ω, f_Ω) , is said to be a **universal cover** over X for the qc fundamental group $\pi_1^{qc}(X; \Omega)$.

10.5. Proof of Theorem 10.7. Now we can prove the main result above on the qc fundamental group in §10.4.

Proof. (Proof of Theorem 10.7) We will proceed in several steps.

Step 1. By *Theorem 7.4* we have

$$\begin{aligned} & Gal(\Omega/k(X)) \\ & \cong \lim_{\leftarrow} Z \in [X; \Omega]_{qc} Gal(k(Z)/k(X)) \\ & \cong \lim_{\leftarrow} Z \in [X; \Omega]_{qc} Aut(Z/X) \\ & = \pi_1^{qc}(X; \Omega) \end{aligned}$$

according to preliminary facts on Galois groups.

Step 2. Fixed a geometric point s of X over Ω . Let $[X; s]_{et}$ be the set of finite étale Galois covers of X over the geometric point s . For any $Z_1, Z_2 \in [X; s]_{et}$ we say

$$Z_1 \leq Z_2$$

if and only if Z_2 is a finite étale Galois cover of Z_1 . Then $[X; s]_{et}$ is a partially ordered set. Put

$$[X; s]_{qc} \triangleq [X; \Omega]_{qc} \bigcap [X; s]_{et}.$$

Let $Z_1, Z_2 \in [X; s]_{qc}$. It is easily seen that Z_2 is a finite étale Galois cover of Z_1 if and only if Z_2 is qc over Z_1 .

It follows that $[X; s]_{qc}$ is a co-final directed subset in $[X; s]_{et}$.

Step 3. Now consider the universal covers X_Ω and X_{et} of X for the groups $\pi_1^{qc}(X; \Omega)$ and $\pi_1^{et}(X; s)$, respectively. From *Step 7* in §10.2 we have

$$\begin{aligned} & Gal(k(X_{et})/k(X)) \\ & \cong \lim_{\leftarrow Z \in [X; s]_{et}} Gal(k(Z)/k(X)) \\ & \cong \lim_{\leftarrow Z \in [X; s]_{et}} Aut(Z/X) \\ & \cong \pi_1^{et}(X; s). \end{aligned}$$

By *Step 2* we have

$$\begin{aligned} & Gal(k(X_{et})/k(X)) \\ & \cong \lim_{\leftarrow Z \in [X; s]_{et}} Gal(k(Z)/k(X)) \\ & \cong \lim_{\leftarrow Z \in [X; s]_{qc}} Gal(k(Z)/k(X)) \end{aligned}$$

since $[X; s]_{qc}$ is co-final in $[X; s]_{et}$.

On the other hand, we have

$$k(X_\Omega) = \lim_{\longrightarrow Z \in [X; \Omega]_{qc}} k(Z)$$

and

$$k(X_{et}) = \lim_{\longrightarrow Z \in [X; s]_{et}} k(Z) = \lim_{\longrightarrow Z \in [X; s]_{qc}} k(Z)$$

as direct limits of direct systems of groups for the function fields.

It is seen that

$$\lim_{\longrightarrow Z \in [X; \Omega]_{qc}} k(Z)$$

is an extension of the field

$$\lim_{\longrightarrow Z \in [X; s]_{qc}} k(Z).$$

It follows that

$$k(X)^{au} = k(X_{et})$$

is a subfield of

$$\Omega = k(X_\Omega).$$

Then we have a tower of Galois extensions of function fields

$$k(X) \subseteq k(X)^{au} \subseteq \Omega$$

from *Lemma 10.1*.

It is seen that $\pi_1^{et}(X; s)$ is isomorphic to the normal subgroup

$$Gal(k(X)^{au}/k(X))$$

of the group $Gal(\Omega/k(X))$. Hence, $\pi_1^{et}(X; s)$ is isomorphic to a normal subgroup of $\pi_1^{qc}(X; \Omega)$ since by *Step 1* we have

$$Gal(\Omega/k(X)) \cong \pi_1^{qc}(X; \Omega).$$

This completes the proof. \square

11. MONODROMY ACTIONS

Naturally there exist three types of monodromy actions for a given integral K -variety, as we have done for arithmetic varieties in [9]:

- Monodromy action of étale fundamental group on the universal cover;
- Monodromy action of absolute Galois group on the sp -completion;
- Monodromy action of ramified group on the sp -completion.

11.1. Monodromy actions of étale fundamental groups. From *Lemma 10.1* and *Theorem 10.2* it is seen there is the below preliminary fact on the étale fundamental group of an integral variety.

Lemma 11.1. *For an integral K -variety X , there is an isomorphism*

$$Aut(X_{et}/X) \cong \pi_1^{et}(X)$$

between groups, where (X_{et}, p_X) is a universal cover of X for the étale fundamental group $\pi_1^{et}(X)$.

Now let X and Y be two integral K -varieties. Suppose that (X_{et}, p_X) and (Y_{et}, p_Y) are universal covers of X and Y for the étale fundamental groups $\pi_1^{et}(X)$ and $\pi_1^{et}(Y)$, respectively.

By *Lemma 11.1* it is seen that each group homomorphism

$$\sigma : \pi_1^{et}(X) \rightarrow \pi_1^{et}(Y).$$

gives a group homomorphism, namely

$$\sigma : Aut(X_{et}/X) \rightarrow Aut(Y_{et}/Y).$$

The converse is true.

Here is the monodromy action of étale fundamental groups on the universal covers.

Lemma 11.2. *Assume that there is a group homomorphism*

$$\sigma : Aut(X_{et}/X) \rightarrow Aut(Y_{et}/Y).$$

Then there is a bijection

$$\tau : Hom(X, Y) \rightarrow Hom(X_{et}, Y_{et}), f \mapsto f_{et}$$

between sets given in a canonical manner:

- Let $f \in \text{Hom}(X, Y)$. Then the map

$$g(x_0) \mapsto \sigma(g)(f(x_0))$$

defines a morphism

$$f_{et} : X_{et} \rightarrow Y_{et}$$

for any $x_0 \in X$ and any $g \in \text{Aut}(X_{et}/X)$.

- Let $f_{et} \in \text{Hom}(X_{et}, Y_{et})$. Then the map

$$p_X(x) \mapsto p_Y(f_{et}(x))$$

defines a morphism

$$f : X \rightarrow Y$$

for any $x \in X_{et}$.

In particular, we have

$$f \circ p_X = p_Y \circ f_{et}.$$

Proof. It is immediate from *Lemma 10.1* and *Theorem 10.2*. □

11.2. Monodromy actions of absolute Galois groups. Let X and Y be two integral K -varieties. Consider the sp -completions (X_{sp}, λ_X) and (Y_{sp}, λ_Y) and the universal covers (X_{et}, p_X) and (Y_{et}, p_Y) for $\pi_1^{et}(X)$ and $\pi_1^{et}(Y)$, respectively.

It is easily seen that there are isomorphisms

$$\text{Gal}(k(X)^{sep}/k(X)) \cong \text{Aut}(X_{sp}/X);$$

$$\text{Gal}(k(Y)^{sep}/k(Y)) \cong \text{Aut}(Y_{sp}/Y).$$

between groups from *Lemma 7.2* and *Theorem 8.7*.

Here is the monodromy action of absolute Galois groups on the sp -completions.

Lemma 11.3. *Suppose that there is a group homomorphism*

$$\sigma : \text{Aut}(X_{sp}/X) \rightarrow \text{Aut}(Y_{sp}/Y).$$

Then there is a bijection

$$\tau : \text{Hom}(X, Y) \rightarrow \text{Hom}(X_{sp}, Y_{sp}), f \mapsto f_{sp}$$

between sets given in a canonical manner:

- Let $f \in \text{Hom}(X, Y)$. Then the map

$$g(x_0) \mapsto \sigma(g)(f(x_0))$$

defines a morphism

$$f_{sp} : X_{sp} \rightarrow Y_{sp}$$

for any $x_0 \in X$ and any $g \in \text{Aut}(X_{sp}/X)$.

- Let $f_{sp} \in \text{Hom}(X_{sp}, Y_{sp})$. Then the map

$$\lambda_X(x) \mapsto \lambda_Y(f_{sp}(x))$$

defines a morphism

$$f : X \rightarrow Y$$

for any $x \in X_{sp}$.

In particular, we have

$$f \circ \lambda_X = \lambda_Y \circ f_{sp}.$$

Proof. It is immediate from Lemma 7.2 and Theorem 8.7. \square

11.3. Monodromy actions of ramified groups. To start with, let's prove a preparatory lemma.

Lemma 11.4. *Let X be an integral K -variety. Suppose that (X_{sp}, λ_X) is an sp -completion of X and (X_{et}, p_X) is a universal cover of X for the étale fundamental group $\pi_1^{et}(X)$. Then there exists canonically a surjective morphism $q_X : X_{sp} \rightarrow X_{et}$ satisfying the below properties:*

- q_X is affine;
- $\lambda_X = p_X \circ q_X$;
- X_{sp} is qc over X_{et} by q_X ;
- (X_{sp}, q_X) is an sp -completion of X_{et} .

In particular, we have $q_X = \lambda_{X_{et}}$.

Proof. Repeat the universal construction for an sp -completion of the integral scheme X_{et} in §8. Then we have a morphism

$$q_X = \lambda_{X_{et}} : X_{sp} \rightarrow X_{et}$$

such that $(X_{sp}, \lambda_{X_{et}})$ is an sp -completion of X_{et} . \square

Let X and Y be two integral K -varieties. Consider the sp -completions (X_{sp}, λ_X) and (Y_{sp}, λ_Y) and the universal covers (X_{et}, p_X) and (Y_{et}, p_Y) for $\pi_1^{et}(X)$ and $\pi_1^{et}(Y)$, respectively.

From Remark 10.9 we have the ramified groups

$$\pi_1^{br}(X) = \frac{\text{Gal}(k(X)^{sep}/k(X))}{\text{Gal}(k(X)^{au}/k(X))};$$

$$\pi_1^{br}(Y) = \frac{Gal(k(Y)^{sep}/k(Y))}{Gal(k(Y)^{au}/k(Y))}.$$

It follows that we have the following result.

Lemma 11.5. *For any integral K -varieties X and Y , there are group isomorphisms*

$$\pi_1^{br}(X) \cong Gal(k(X)^{sep}/k(X)^{au}) \cong Aut(X_{sp}/X_{et});$$

$$\pi_1^{br}(Y) \cong Gal(k(Y)^{sep}/k(Y)^{au}) \cong Aut(Y_{sp}/Y_{et}).$$

Here is the monodromy action of ramified groups on the sp -completions. It will play an important role in the anabelian geometry.

Lemma 11.6. *Suppose that there is a group homomorphism*

$$\sigma : Aut(X_{sp}/X_{et}) \rightarrow Aut(Y_{sp}/Y_{et}).$$

Then there is a bijection

$$\tau : Hom(X_{et}, Y_{et}) \rightarrow Hom(X_{sp}, Y_{sp}), f \mapsto f_{sp}$$

between sets given in a canonical manner:

- *Let $f \in Hom(X_{et}, Y_{et})$. Then the map*

$$g(x_0) \mapsto \sigma(g)(f(x_0))$$

defines a morphism

$$f_{sp} : X_{sp} \rightarrow Y_{sp}$$

for any $x_0 \in X_{et}$ and any $g \in Aut(X_{sp}/X_{et})$.

- *Let $f_{sp} \in Hom(X_{sp}, Y_{sp})$. Then the map*

$$\lambda_{X_{et}}(x) \mapsto \lambda_{Y_{et}}(f_{sp}(x))$$

defines a morphism

$$f : X_{et} \rightarrow Y_{et}$$

for any $x \in X_{sp}$.

In particular, we have

$$f \circ \lambda_{X_{et}} = \lambda_{Y_{et}} \circ f_{sp}.$$

Proof. It is immediate from Lemmas 11.3-4. □

12. PROOF OF THE MAIN THEOREM

12.1. Preliminary lemmas. Let X and Y be two integral K -varieties. Fixed any sp -completions (X_{sp}, λ_X) and (Y_{sp}, λ_Y) of X and Y , and any universal covers (X_{et}, p_X) and (Y_{et}, p_Y) of X and Y for the groups $\pi_1^{et}(X)$ and $\pi_1^{et}(Y)$, respectively.

There are several results on the sp -completions and the universal covers of X and Y , respectively (c.f. [9]).

Remark 12.1. From a viewpoint of sp -completion, it is seen that arithmetic varieties and integral K -varieties are very different. In fact, for arithmetic varieties Z_1, Z_2 , by *Theorem 9.15* we have

$$k(Z_1) = k(Z_2) \implies Sp[Z_1] = Sp[Z_2].$$

However, for integral K -varieties Z_1, Z_2 , from *Lemma 8.9* it is seen that

$$k(Z_1) = k(Z_2) \implies Sp[Z_1] = Sp[Z_2]$$

does not hold in general.

Lemma 12.2. *Suppose $k(X) = k(Y)$ and $Sp[X] = Sp[Y]$. Then there exists a bijection τ from $Hom(X, Y)$ onto $Hom(X_{sp}, Y_{sp})$ given in a canonical manner. In particular, $Hom(X, Y)$ must be a non-void set.*

Proof. Fixed any sp -completions X_{sp} and Y_{sp} of X and Y , respectively. By *Lemma 8.8* there is an isomorphism

$$f_{sp} : X_{sp} \rightarrow Y_{sp}$$

according to the assumption that $Sp[X] = Sp[Y]$ holds.

By *Theorem 8.7* we have

$$Aut(X_{sp}/X) \cong Gal(k(X)^{sep}/k(X)) \cong Aut(Y_{sp}/Y).$$

From *Lemma 11.3* we immediately obtain the desired properties. \square

Remark 12.3. From a viewpoint of graph functor Γ , it is seen that arithmetic varieties and integral K -varieties are also very different. In fact, for arithmetic varieties Z_1, Z_2 , we have

$$k(Z_1) \subseteq k(Z_2) \implies \Gamma(Z_1) \subseteq \Gamma(Z_2).$$

However, for integral K -varieties Z_1, Z_2 , in general, it is not true that

$$k(Z_1) \subseteq k(Z_2) \implies \Gamma(Z_1) \subseteq \Gamma(Z_2)$$

holds.

Lemma 12.4. *Let $k(X)$ be separably generated over $k(Y)$. Then there are the following statements:*

- There is a homomorphism

$$\sigma_{sp} : \text{Gal}(k(X)^{sep}/k(X)) \rightarrow \text{Gal}(k(Y)^{sep}/k(Y)).$$

- There is a bijection τ from $\text{Hom}(X, Y)$ onto $\text{Hom}(X_{sp}, Y_{sp})$ given in a canonical manner. In particular, $\text{Hom}(X, Y)$ is empty if and only if so is $\text{Hom}(X_{sp}, Y_{sp})$.
- Let $\Gamma(X_{sp}) \supseteq \Gamma(Y_{sp})$. Then $\text{Hom}(X, Y)$ must be a non-void set.

Proof. It suffices to prove the third statement. Suppose that $\Gamma(Y_{sp})$ is a subgraph of $\Gamma(X_{sp})$. Take the sp -completions (X_{sp}, λ_X) and (Y_{sp}, λ_Y) of X and Y , respectively.

It is seen that the ring B of an affine open set V in Y_{sp} must be embedded into the ring A of some certain affine open set U in X_{sp} as a subring. In deed, choose A to be the ring over B generated by the set

$$\Delta_B = \{\sigma(w) : w \in \Delta, \sigma \in \text{Gal}(k(X)^{sep}/k(Y)^{sep})\}$$

where $\Delta \subseteq k(X)^{sep} \setminus k(Y)^{sep}$ is a set of generators of $k(X)^{sep}$ over $k(Y)^{sep}$. It is easily seen that $U = \text{Spec}(A)$ is an affine open set in an sp -completion of X that is essentially equal to the given X_{sp} from the universal construction for X_{sp} in §8.2.

Conversely, each A must contain some B . In fact, let ξ and η be the generic points of X and Y , respectively. Take any point y_0 in V . We have the specializations

$$\eta \rightarrow y_0 \text{ in } Y_{sp};$$

$$\xi \rightarrow y_0 \text{ in } X_{sp}.$$

From *Lemma 8.2* we have an affine open set $U = \text{Spec}(A)$ in X_{sp} containing ξ and y_0 ; then, U also contains η . Hence, A contains some B such that $y_0 \in V = \text{Spec}(B)$.

It follows that there is a homomorphism

$$f_U : U = \text{Spec}(A) \rightarrow V = \text{Spec}(B)$$

defined by the inclusion. This gives us a scheme homomorphism

$$f_{sp} : X_{sp} \rightarrow Y_{sp}.$$

By the projections $\lambda_X : X_{sp} \rightarrow X$ and $\lambda_Y : Y_{sp} \rightarrow Y$ we have a unique homomorphism $f : X \rightarrow Y$ satisfying the condition

$$\lambda_{sp} \circ f_{sp} = f \circ \lambda_{sp}.$$

This completes the proof. \square

Lemma 12.5. *Let $k(X)$ be separably generated over $k(Y)$. Then there are the following statements:*

- There is a homomorphism

$$\sigma_{br} : Gal(k(X)^{sep}/k(X)^{au}) \rightarrow Gal(k(Y)^{sep}/k(Y)^{au}).$$
- There is a bijection τ from $Hom(X_{et}, Y_{et})$ onto $Hom(X_{sp}, Y_{sp})$ given in a canonical manner. In particular, $Hom(X_{et}, Y_{et})$ is empty if and only if so is $Hom(X_{sp}, Y_{sp})$.
- Let $\Gamma(X_{sp}) \supseteq \Gamma(Y_{sp})$. Then $Hom(X_{et}, Y_{et})$ must be a non-void set.

Proof. It is immediate from Lemmas 11.4-6, 12.4. \square

Lemma 12.6. Let $k(X)$ be separably generated over $k(Y)$. Then there are the following statements:

- There is a homomorphism

$$\sigma_{et} : Gal(k(X)^{au}/k(X)) \rightarrow Gal(k(Y)^{au}/k(Y)).$$
- There is a bijection τ from $Hom(X, Y)$ onto $Hom(X_{au}, Y_{au})$ given in a canonical manner. In particular, $Hom(X, Y)$ is empty if and only if so is $Hom(X_{au}, Y_{au})$.
- Let $\Gamma(X_{sp}) \supseteq \Gamma(Y_{sp})$. Then $Hom(X_{et}, Y_{et})$ must be a non-void set.

Proof. It is immediate from Lemmas 11.2, 12.4. \square

12.2. Proof of the main theorem. Now we can give the proof of the main theorem in the paper.

Proof. (Proof of Theorem 1.3) Noticed that from Lemma 11.5 we have the ramified groups

$$\begin{aligned}\pi_1^{br}(X) &= \frac{Gal(k(X)^{sep}/k(X))}{Gal(k(X)^{au}/k(X))} \cong Aut(X_{sp}/X_{et}); \\ \pi_1^{br}(Y) &= \frac{Gal(k(Y)^{sep}/k(Y))}{Gal(k(Y)^{au}/k(Y))} \cong Aut(Y_{sp}/Y_{et}).\end{aligned}$$

Then we have

$$\begin{aligned}Hom(X, Y) &\cong Hom(\pi_1^{br}(X), \pi_1^{br}(Y)) \\ &\cong Hom\left(\frac{Aut(X_{sp}/X)}{Aut(X_{et}/X)}, \frac{Aut(Y_{sp}/Y)}{Aut(Y_{et}/Y)}\right)\end{aligned}$$

from Lemmas 11.1-4, 11.6, 12.2, 12.4-6.

This completes the proof. \square

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